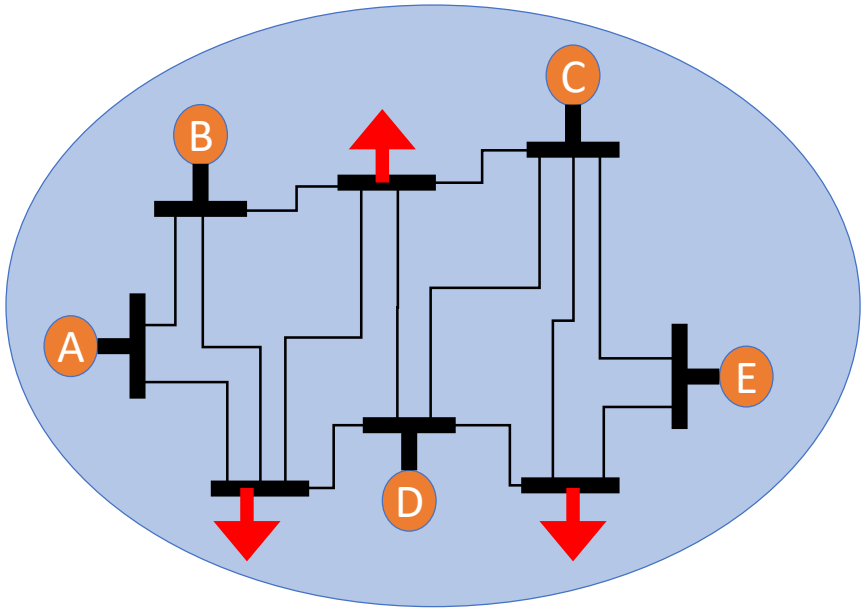
What is power flow analysis?

- Generators inject power at various nodes of the network
- Loads extract power from various nodes of the network
- There are multiple paths between generation and loads
- We don’t control directly how much power flows in each line
- For given power injections and extractions, we must be able to check that the flows and voltages will be acceptable



What is special about power flow analysis?

- Variables in conventional DC circuit analysis:
  - Voltages
  - Currents
- Variables in conventional AC circuit analysis:
  - Magnitude and phase of voltages
  - Magnitude and phase of currents
- Variables in power flow analysis:
  - Magnitude and phase of voltages
  - Active and reactive powers

Why use powers as variables?

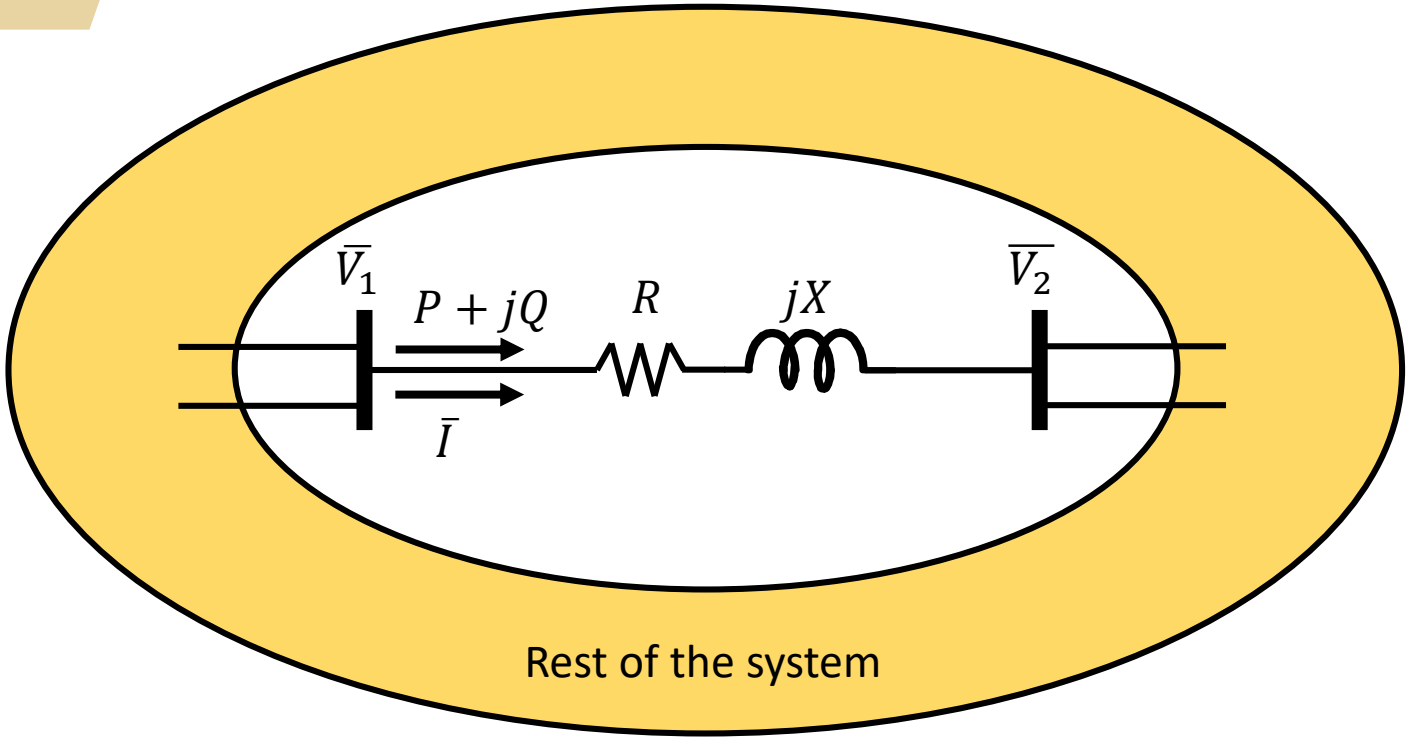
- Generators
  - Convert mechanical power into electrical power
  - Paid to supply power rather than current
- Loads
  - Power is what consumers are buying
  - Easier to add powers than combine loads modeled as parallel impedances
- Consequence:
  - $p(t) = v(t)i(t)$
  - Product of variables → non-linear equations

Overview

- Develop an **intuitive understanding** of the relation between active and reactive power flows and voltages
- Develop the **mathematical formulation** of the power flow problem
- Develop a **solution technique** for the non-linear power flow equations
- Discuss techniques for **controlling flows and voltages**
- Discuss **applications** of power flow computations

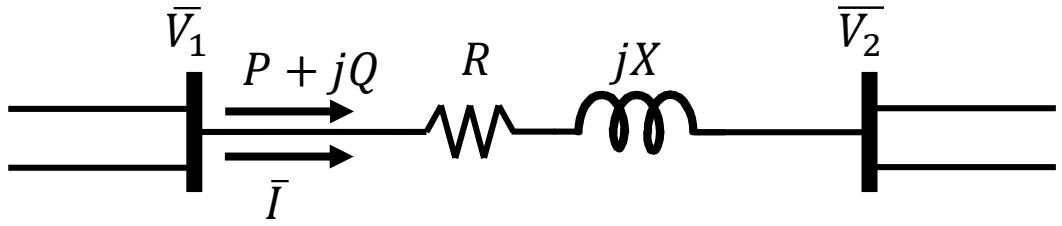
Develop an intuitive understanding of flows and voltages

Intuitive understanding of flows and voltages



- Consider one line somewhere in the system
- $\overline{V_1}$: Sending end voltage
- $\overline{V_2}$: Receiving end voltage
- P, Q: Active and reactive power in the line at the sending end

Powers and voltage drops



$$\bar{S} = P + jQ = \bar{V}_1\,\bar{I}^*$$

$$\bar{I} = \frac{\bar{S}^*}{\bar{V}_1^*} = \frac{P - jQ}{V_1}$$

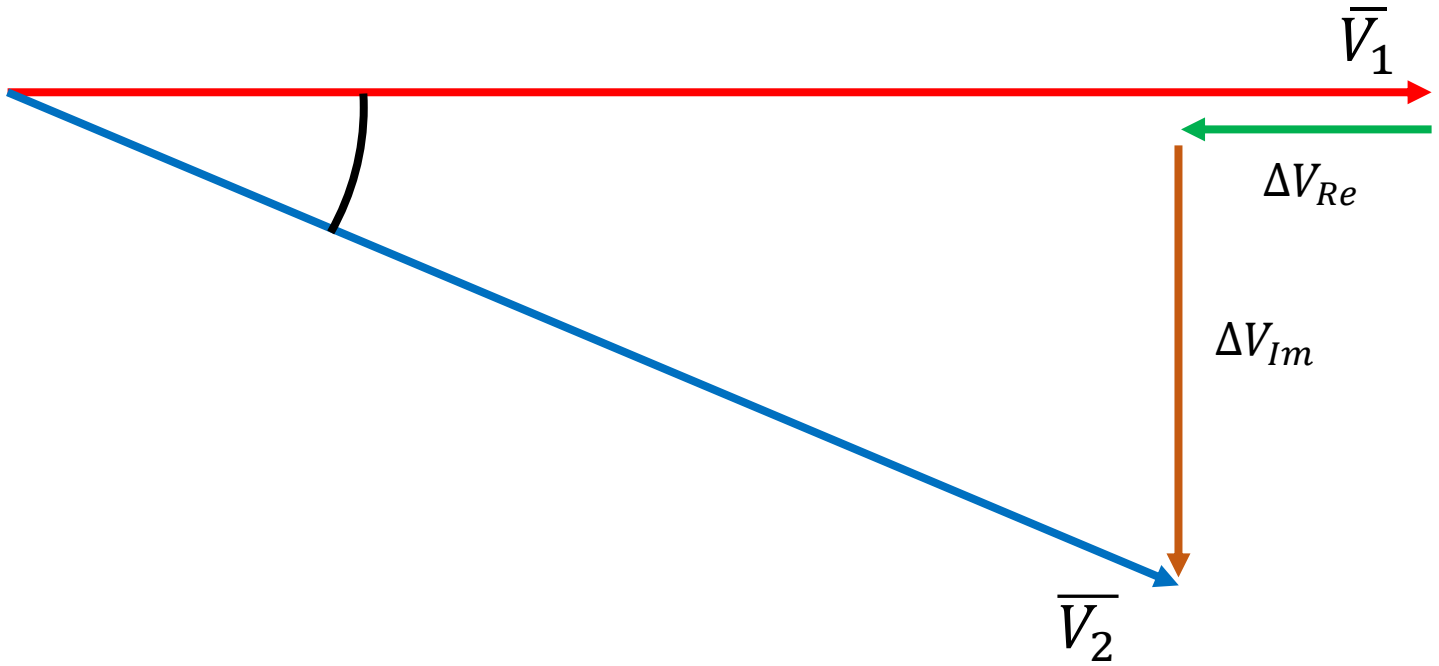
$$\bar{V}_2 = \bar{V}_1 - (R + jX)\bar{I}$$

$$\bar{V}_2 = \bar{V}_1 - \frac{RP + XQ}{V_1} - j\frac{XP - RQ}{V_1}$$

Powers and voltage drops

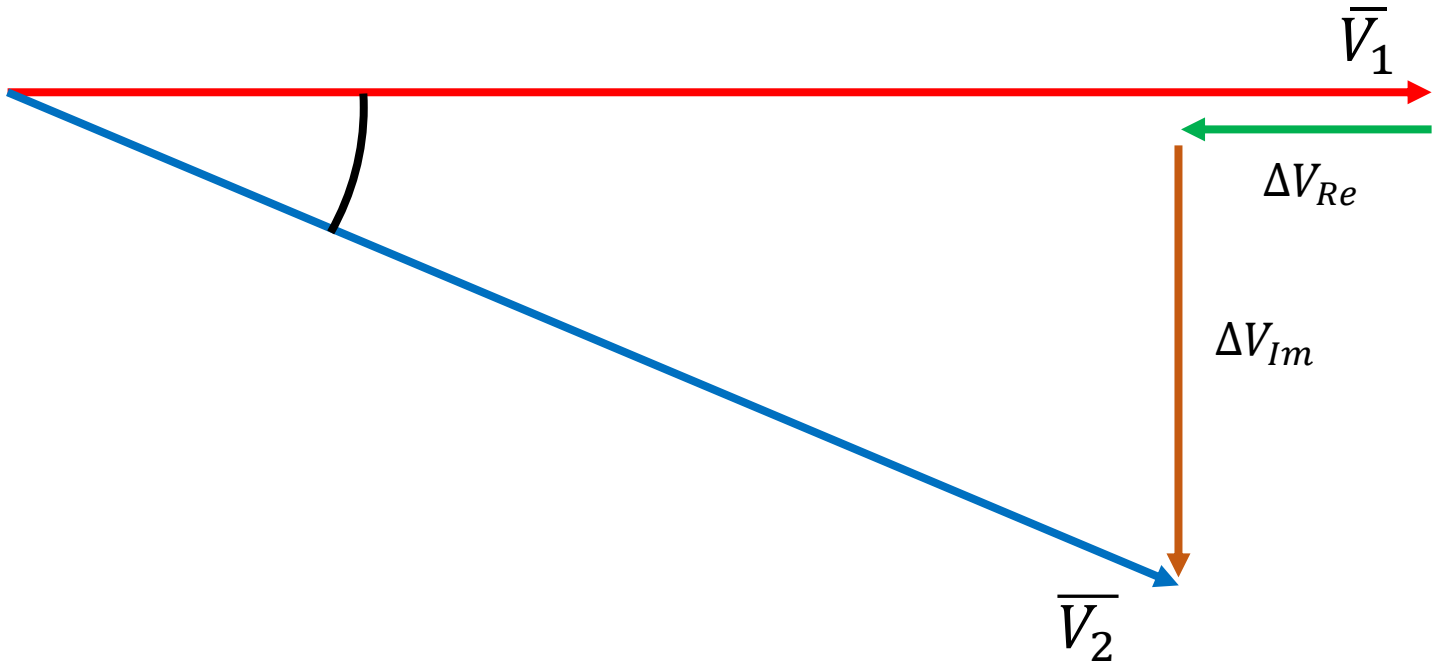
$$\overline{V_2} = \overline{V_1} - \frac{RP + XQ}{V_1} - j\frac{XP - RQ}{V_1}$$

$$\overline{V_2} = \overline{V_1} - \Delta V_{Re} - j\Delta V_{Im}$$



Powers and voltage drops

- ΔV_{Re} causes most of the voltage difference in magnitude between $\overline{V_1}$ and $\overline{V_2}$
- ΔV_{Im} causes most of the voltage difference in angle between $\overline{V_1}$ and $\overline{V_2}$



Powers and voltage drops

$$\overline{V_2} = \overline{V_1} - \frac{RP + XQ}{V_1} - j\frac{XP - RQ}{V_1}$$

$$\overline{V_2} = \overline{V_1} - \Delta V_{Re} - j\Delta V_{Im}$$

$$\Delta V_{Re} = \frac{RP + XQ}{V_1}$$

$$\Delta V_{Im} = \frac{XP - RQ}{V_1}$$

$X \gg R$ ⇒

$$\Delta V_{Re} \approx \frac{XQ}{V_1}$$

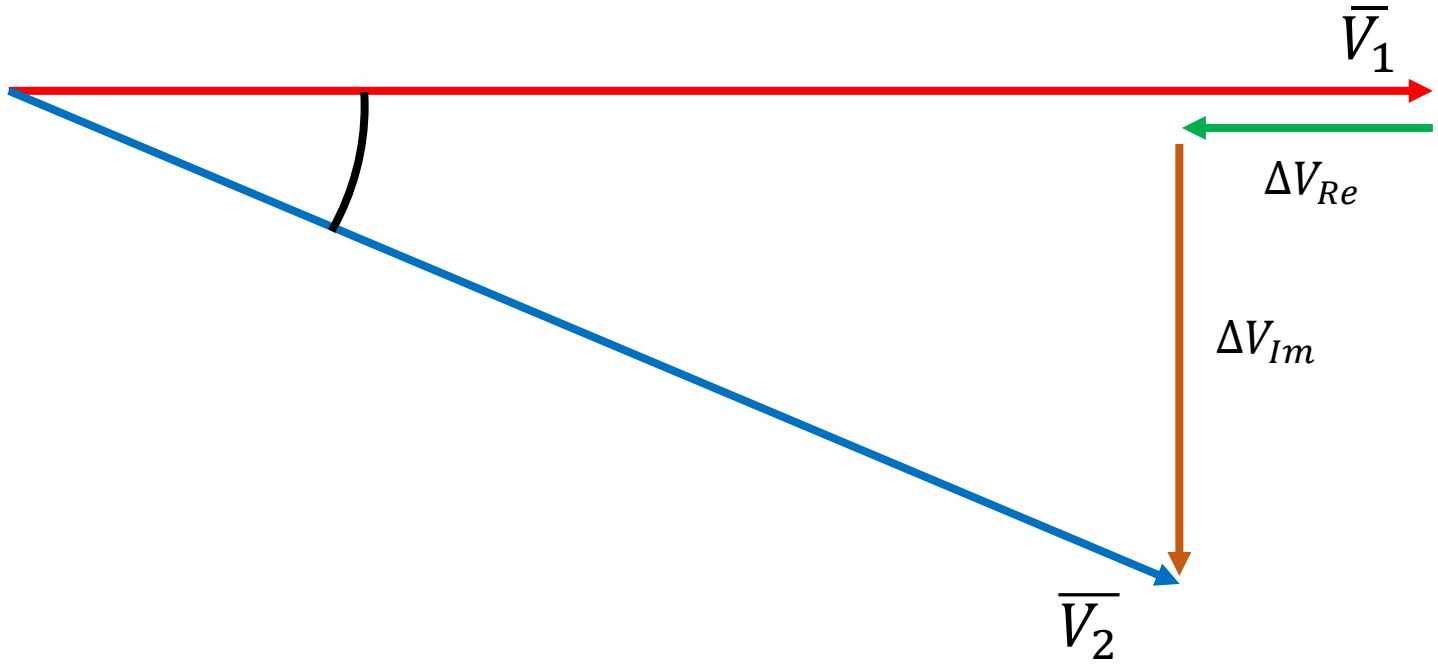
$$\Delta V_{Im} \approx \frac{XP}{V_1}$$

- For typical transmission lines: $X \approx 10\,R$
- For typical distribution lines: $X \approx R$ to $X \approx 2\,R$
  - $X >> R$ is reasonable for transmission but not for distribution

Powers and voltage drops

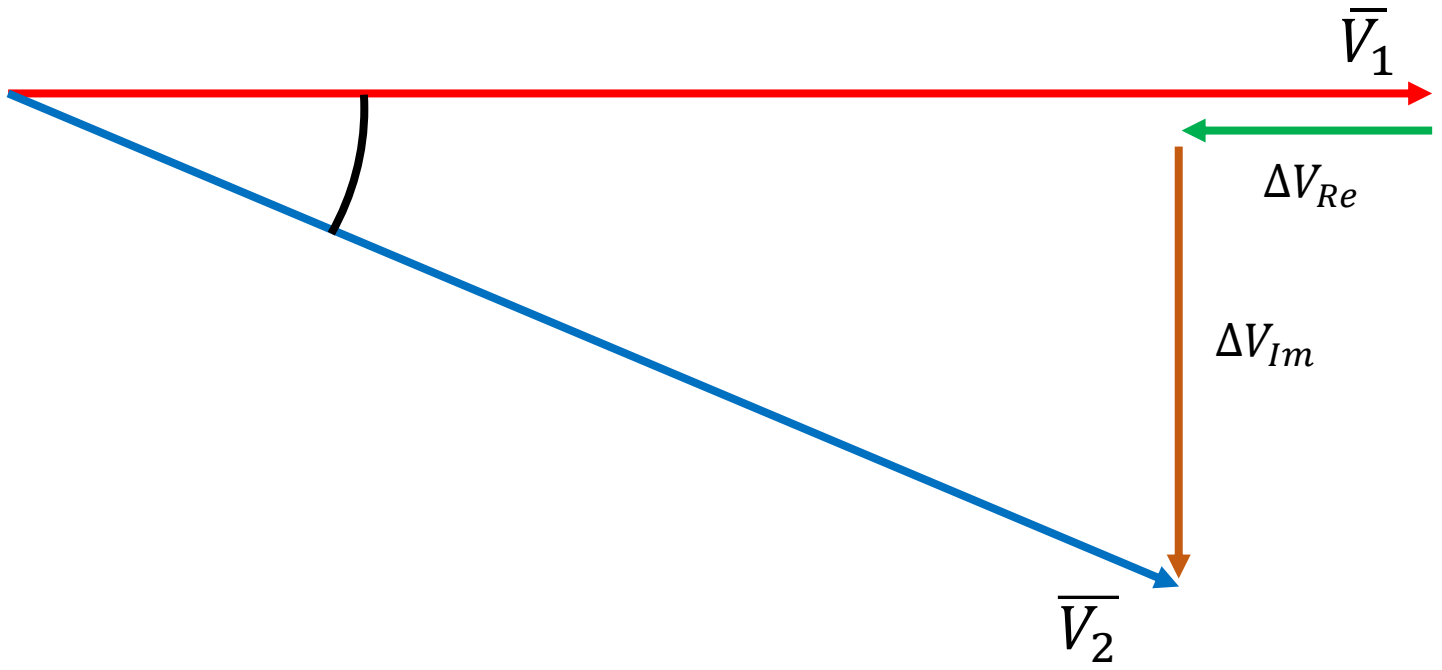
$$\Delta V_{Re} \approx \frac{X}{V_1} Q$$

$$\Delta V_{Im} \approx \frac{X}{V_1} P$$



- The flow of reactive power causes most of the drop in voltage magnitude
- The flow of active power causes most of the phase angle difference

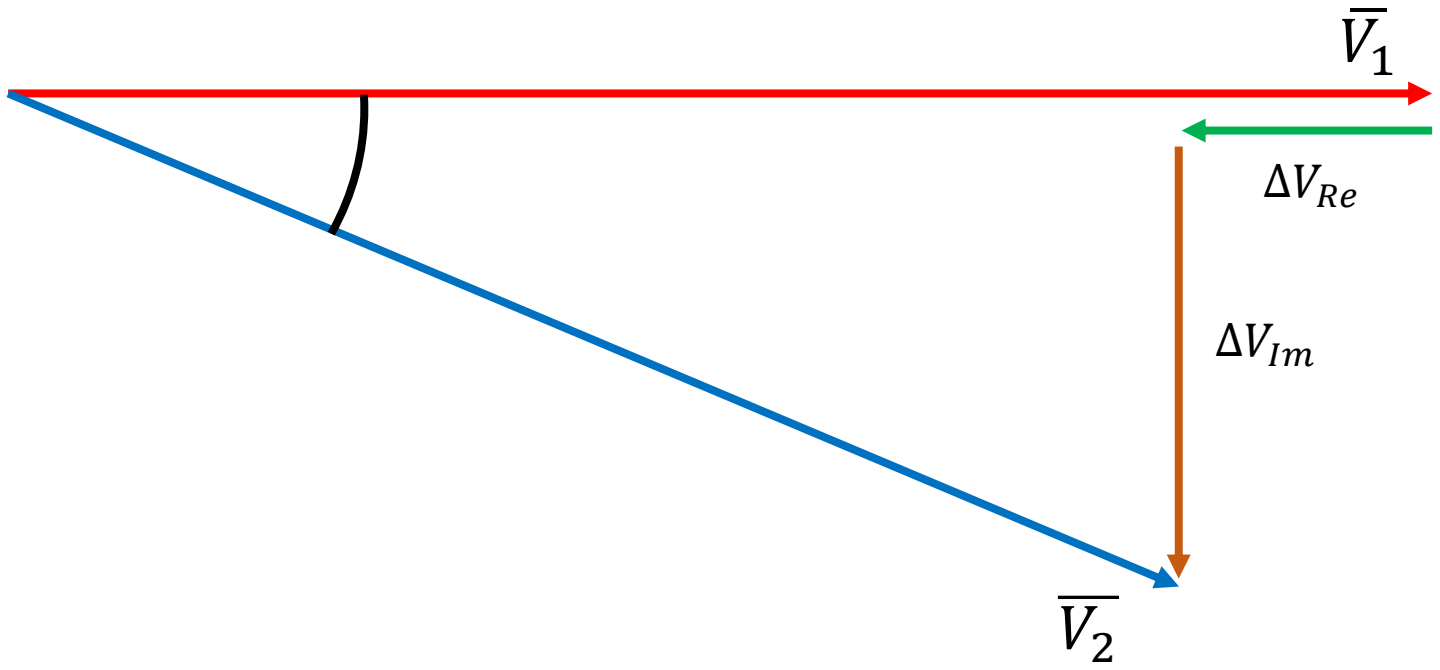
Active power flow and voltage angle



$$\Delta V_{Im} \approx \frac{X}{V_1} P$$

- Large active power flows cause large phase angle differences
- This can cause stability problems

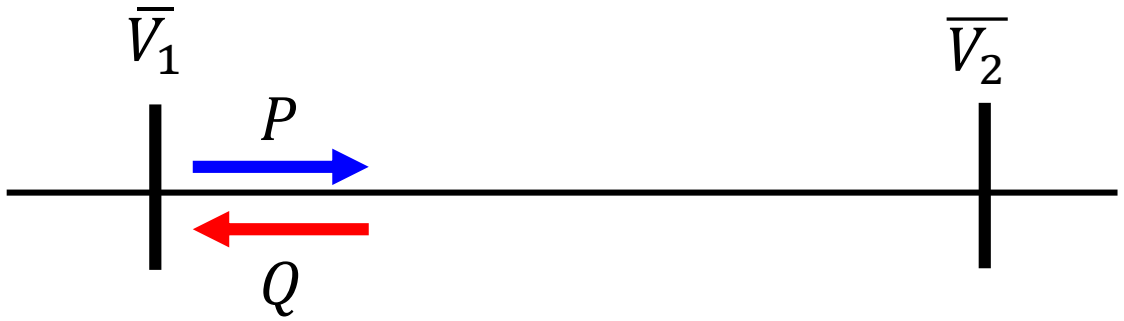
Reactive power flow and voltage magnitude



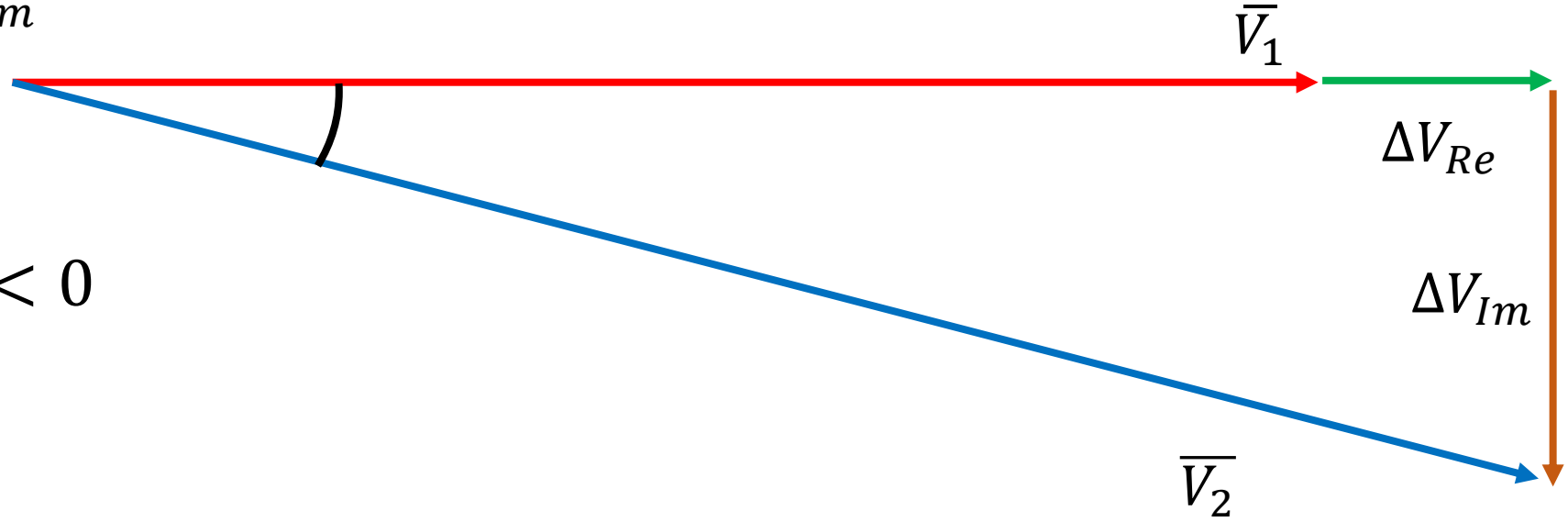
$$\Delta V_{Re} \approx \frac{X}{V_1} Q$$

- Large reactive power flows create large voltage drops
- This can also create stability problems

Active and reactive powers can flow in opposite directions



$$\overline{V_2} = \overline{V_1} - \Delta V_{Re} - j\Delta V_{Im}$$

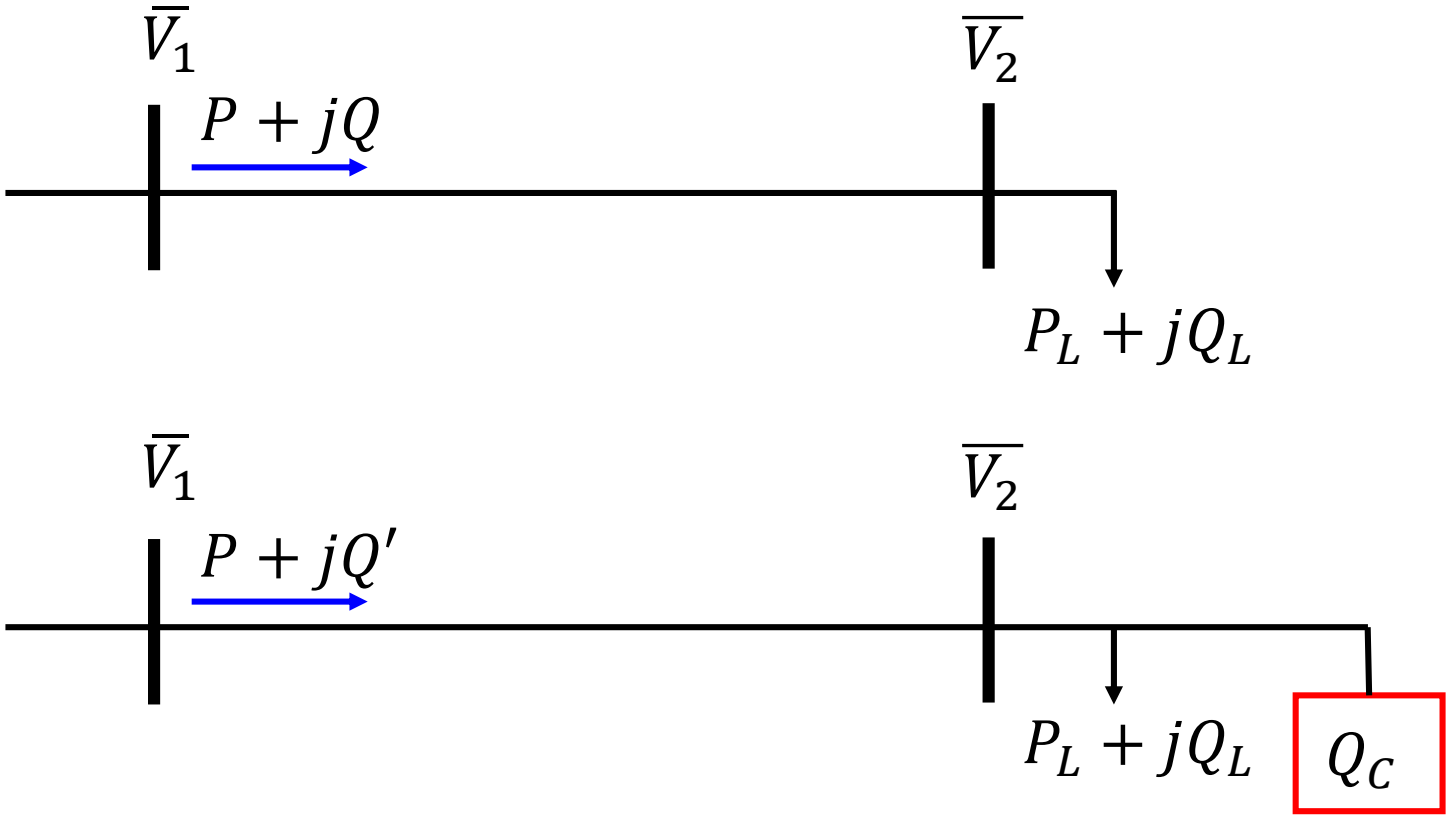


$$\Delta V_{Re} \approx \frac{X}{V_1} Q \text{ with } Q < 0$$

$\overline{V_2}$ lags $\overline{V_1}$

$|\overline{V_2}| > |\overline{V_1}|$

Reactive compensation



Large Q_L → large Q → $|\overline{V_2}| \ll |\overline{V_1}|$

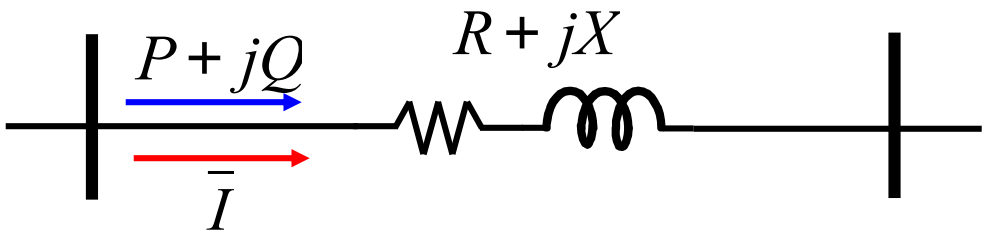
Inductive load: $Q_L > 0$

Reactive compensation: $Q_C < 0$

Q' small → $|\overline{V_2}| \sim |\overline{V_1}|$

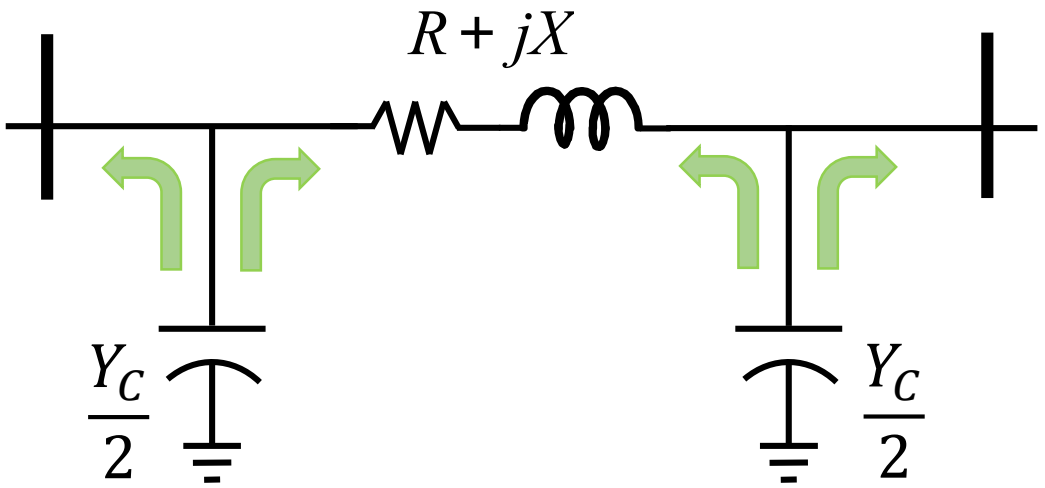
Q_C: capacitor bank, generator or SVC

Reactive power “losses” in lines and transformers



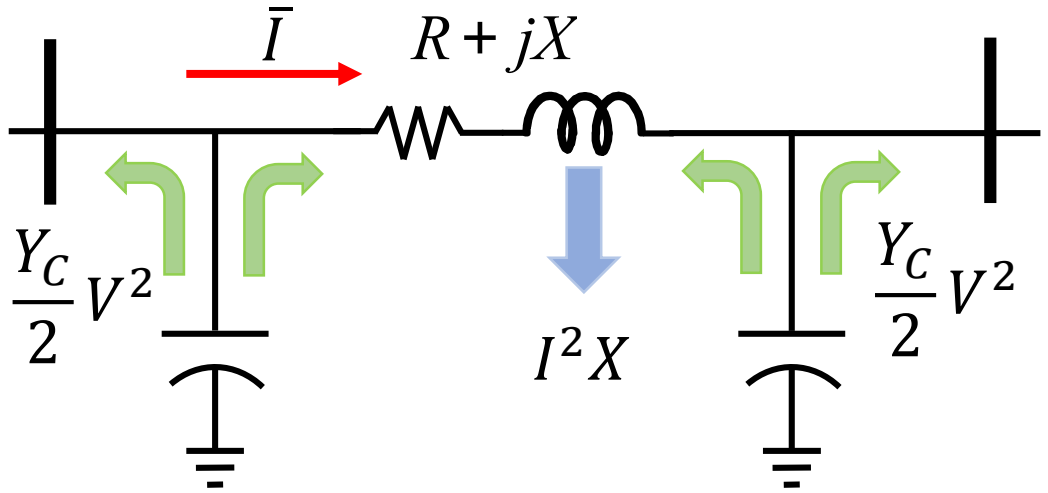
- Active power losses in the line: I^2R
- Reactive power “losses” in the line: I^2X
- These reactive power “losses” do not represent a loss of energy!
- However, this reactive power losses must be supplied by a generator or through reactive power compensation

Reactive power production in lines



- For long Extra High Voltage (EHV) lines, the shunt capacitive reactances of the line model are significant
- They produce reactive power: $\frac{Y_C}{2} V^2$

Reactive power in lines



- Normally $0.95 \leq V \leq 1.05\ p.u.$ → $\frac{Y_C}{2}V^2$ is roughly constant
- Lightly loaded line → I is small → $I^2X < Y_CV^2$ → line produces reactive power
- Heavily loaded line → I is large → $I^2X > Y_CV^2$ → line absorbs reactive power

Nodal Analysis

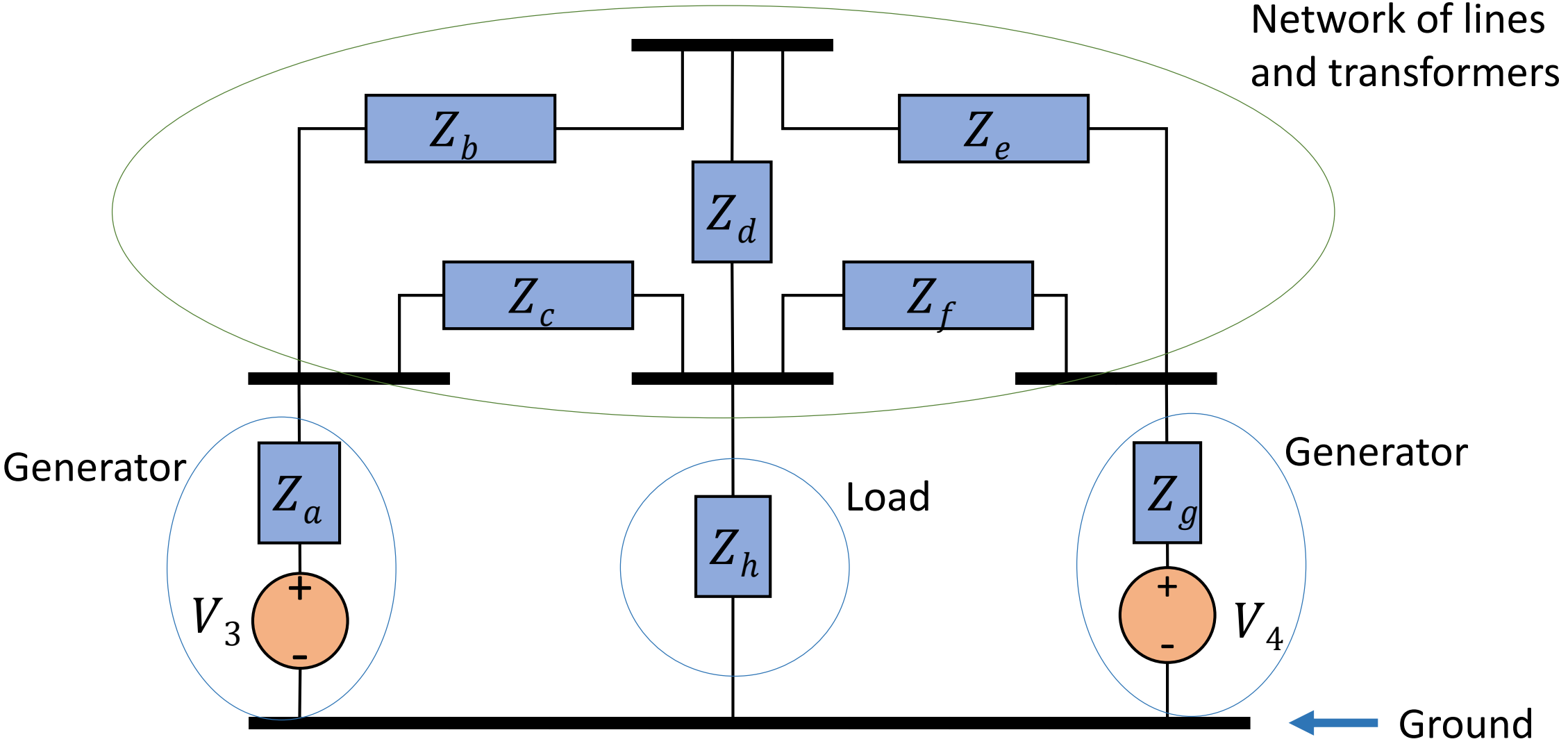
Solving large networks

- Power networks usually involve thousands of nodes and branches
- We need to develop:
  - A systematic method for building the equations
  - An efficient method for solving these equations

→ Nodal analysis

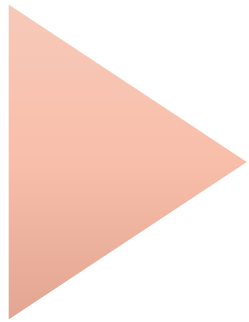
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Nodal analysis using an example



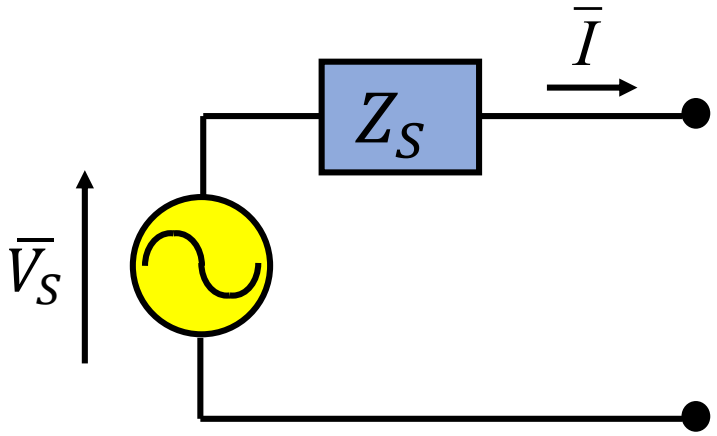
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First solve this example the long way using KCL and Ohm's law

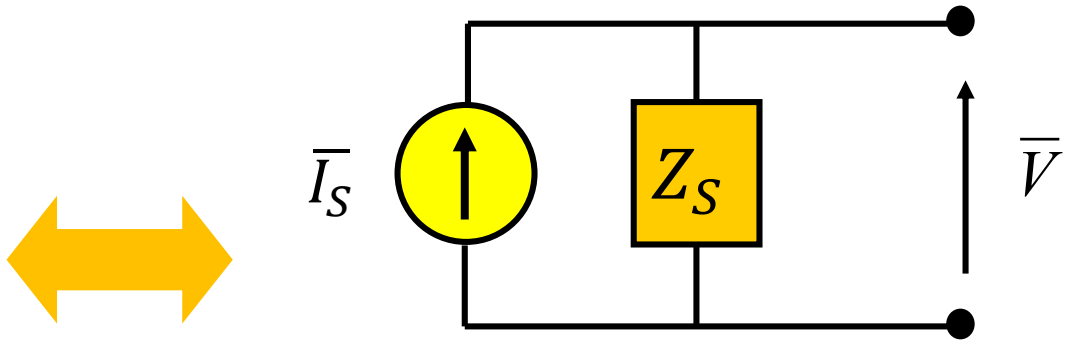


Generalize the results to derive a much simpler technique

Step 1: Convert voltage sources to current sources



Thevenin model

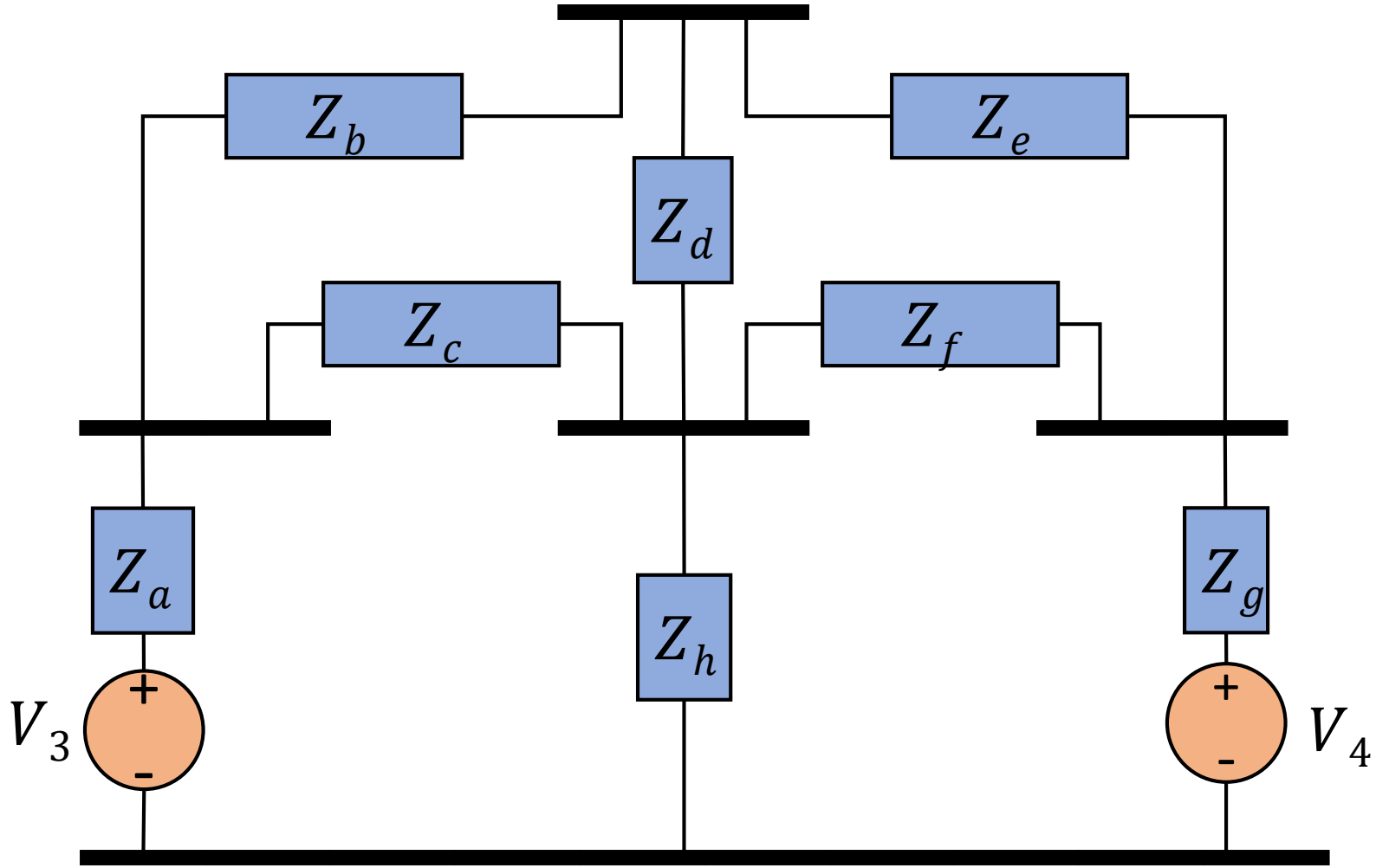


Norton model

$$\bar{I}_S = \frac{\bar{V}_S}{Z_S}$$

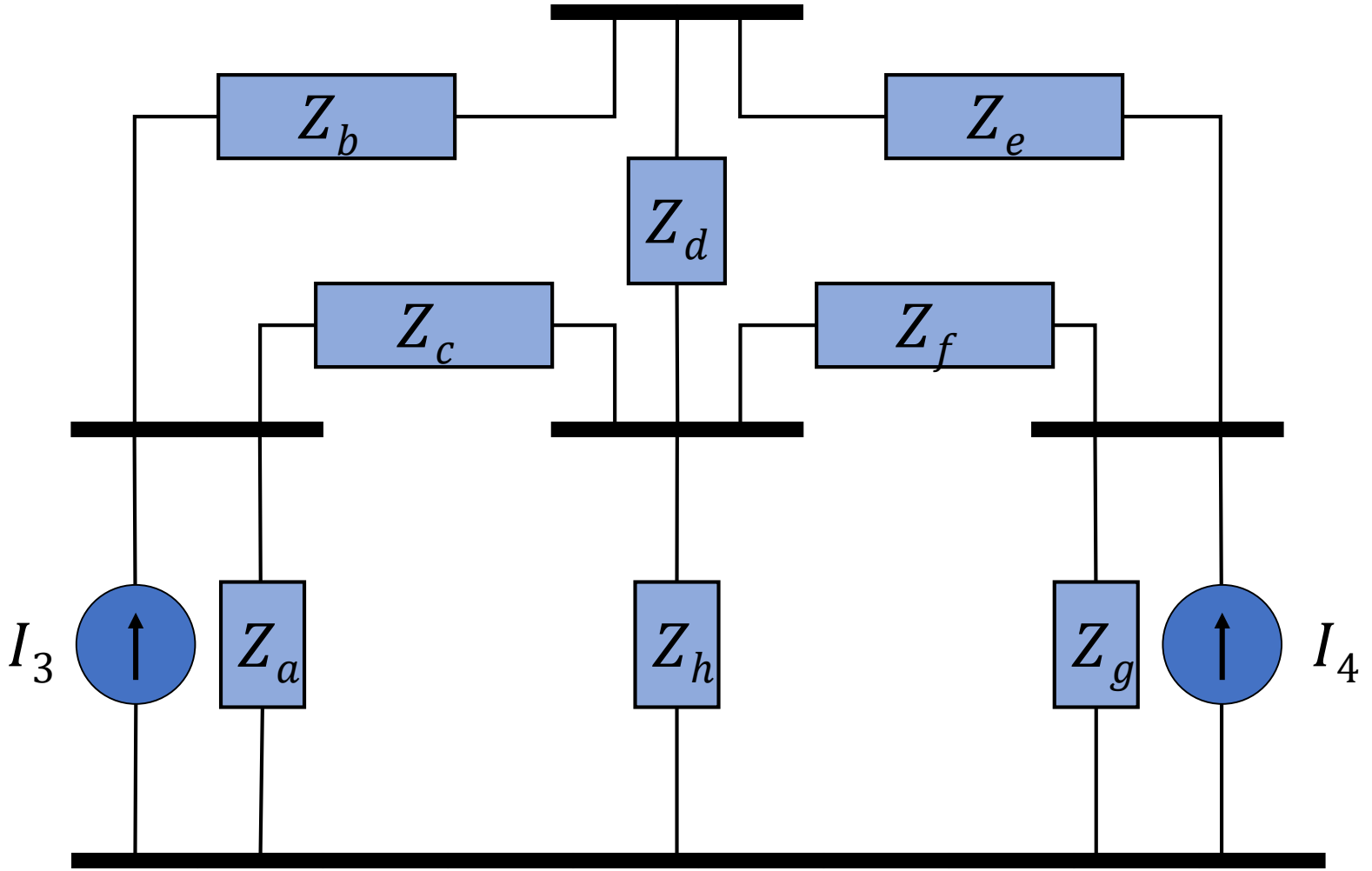
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Step 1: Convert voltage sources to current sources



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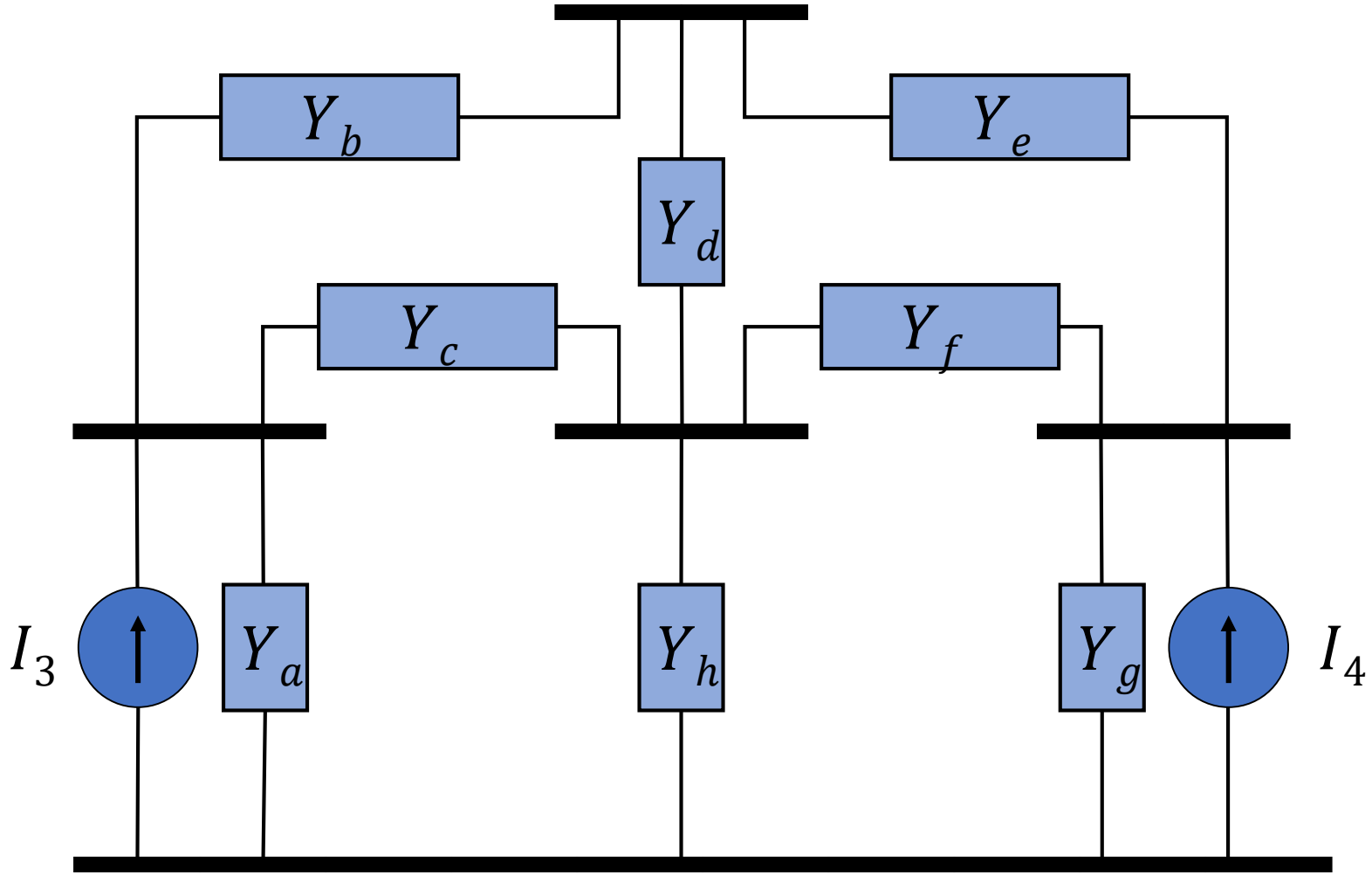
Step 1: Convert voltage sources to current sources



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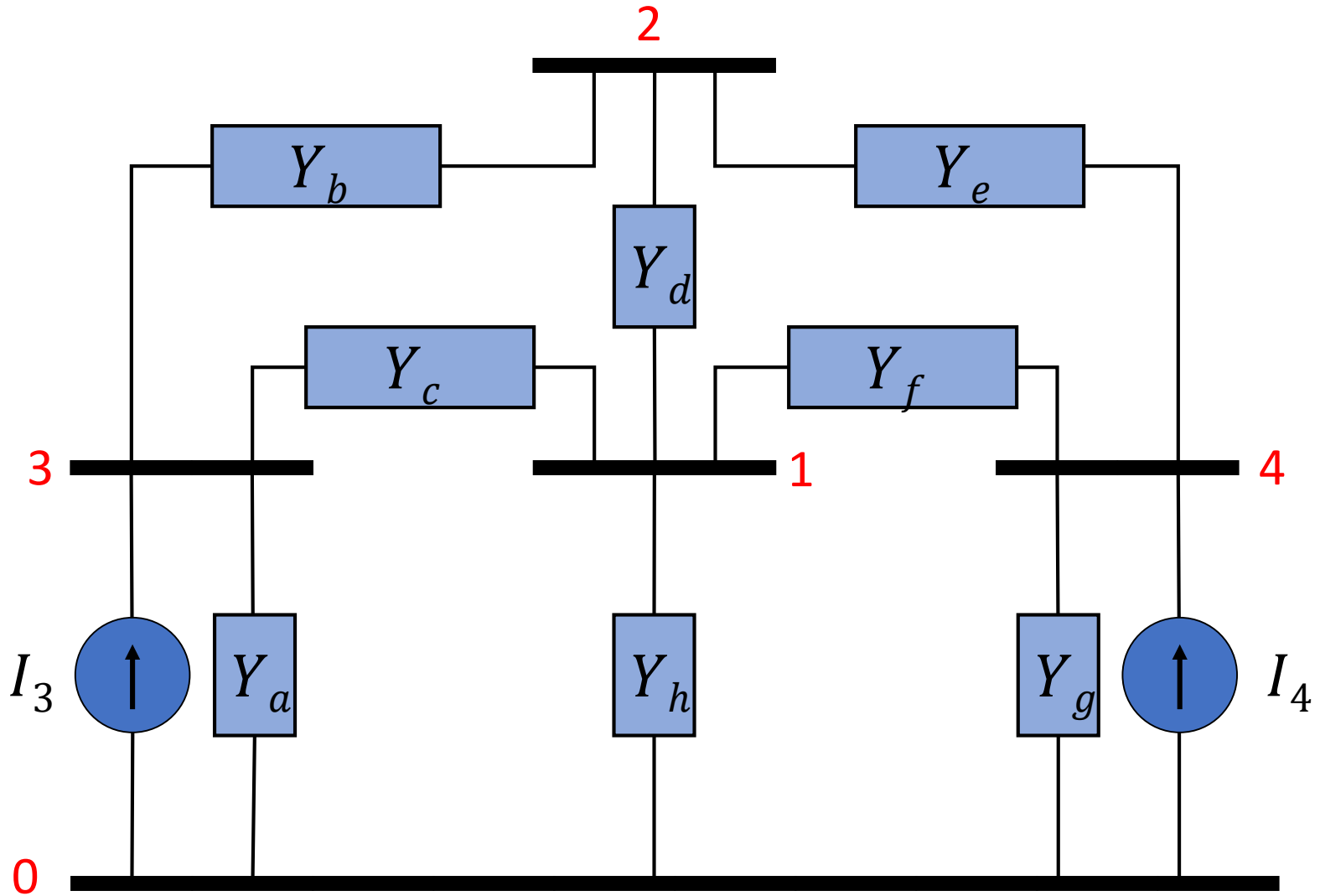
Step 2: Convert all impedances to admittances

$$Y = \frac{1}{Z}$$



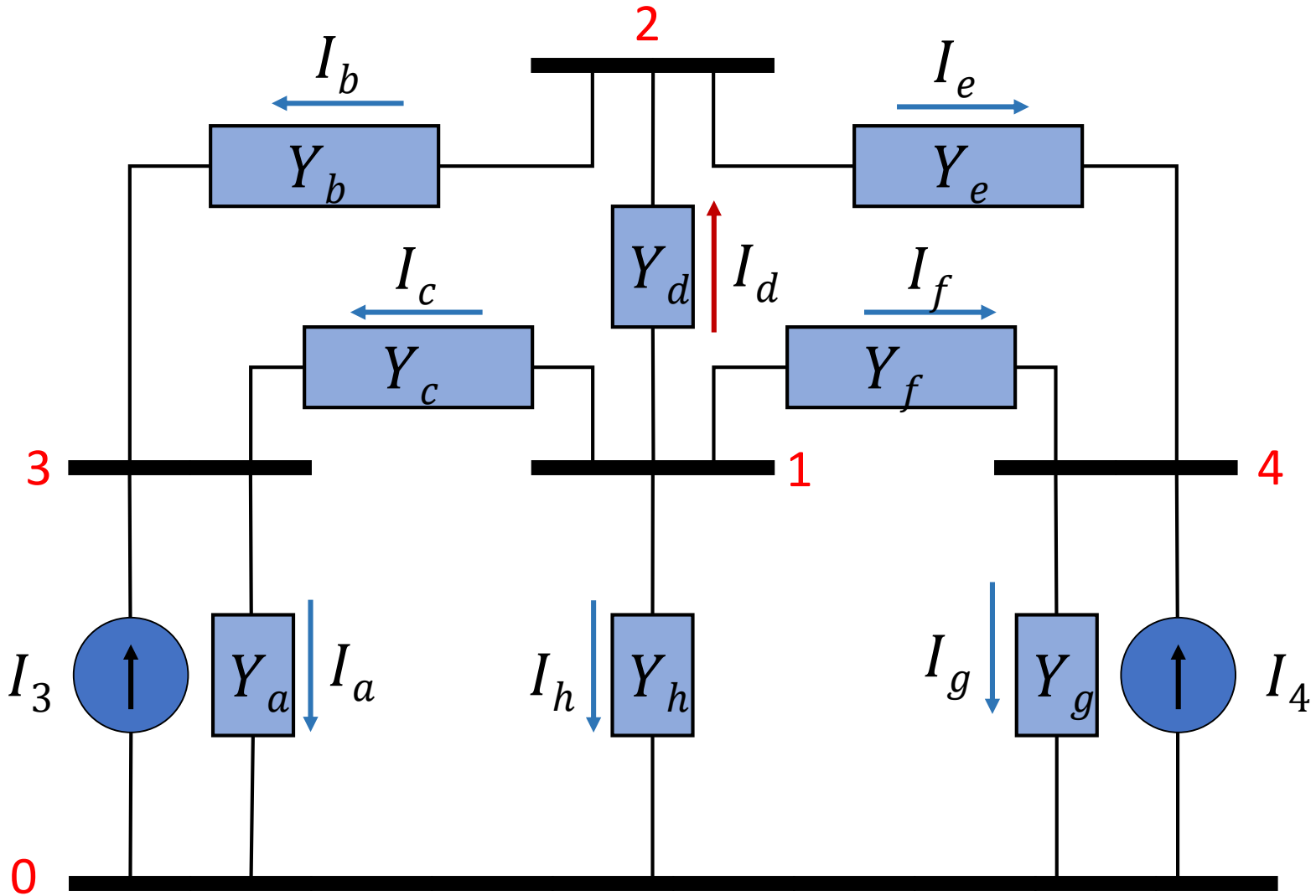
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Step 3: Number the nodes



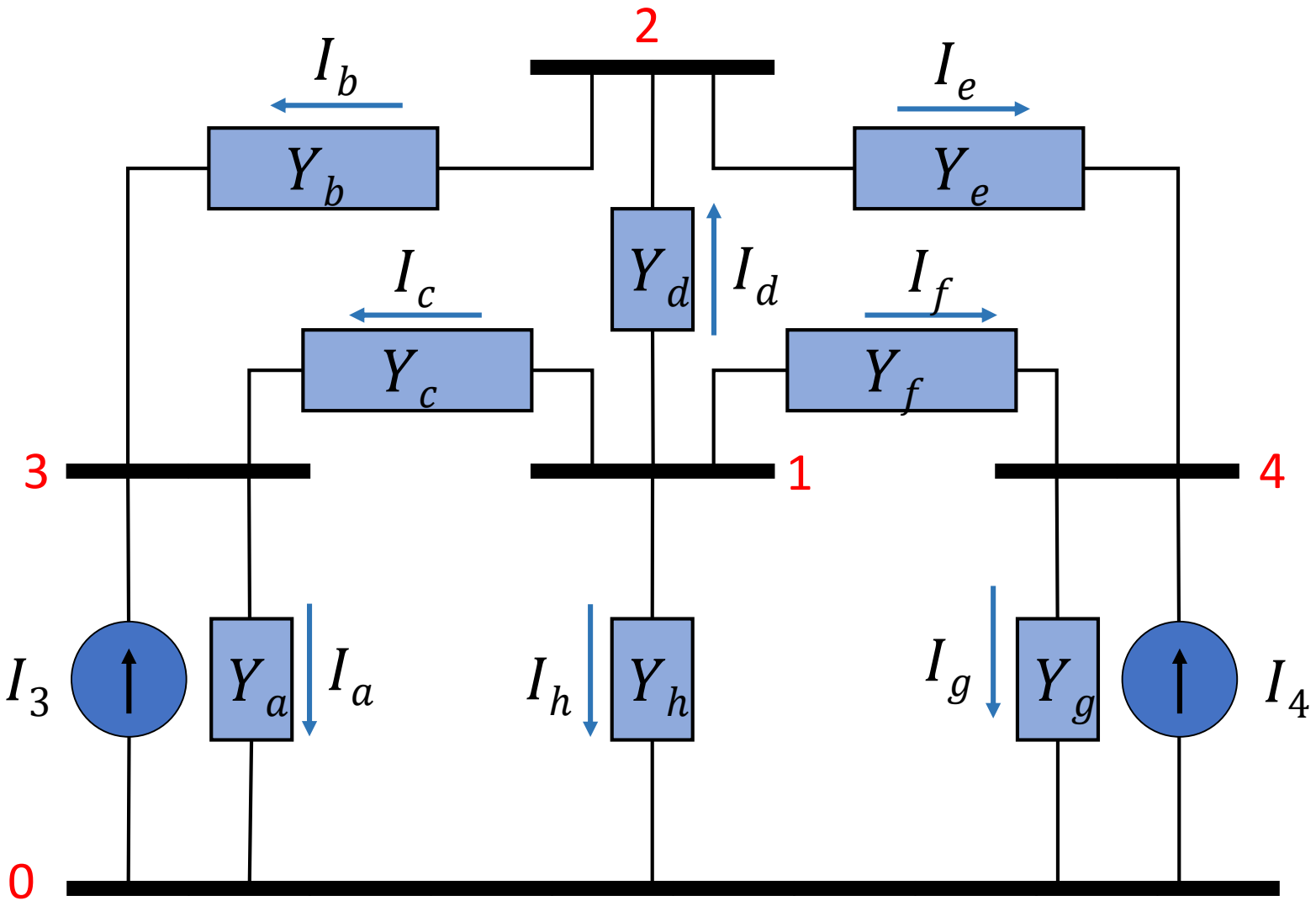
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Step 4: Write KCL at each node



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Step 4: Write KCL at each node



$$I_c + I_d + I_f + I_h = 0$$

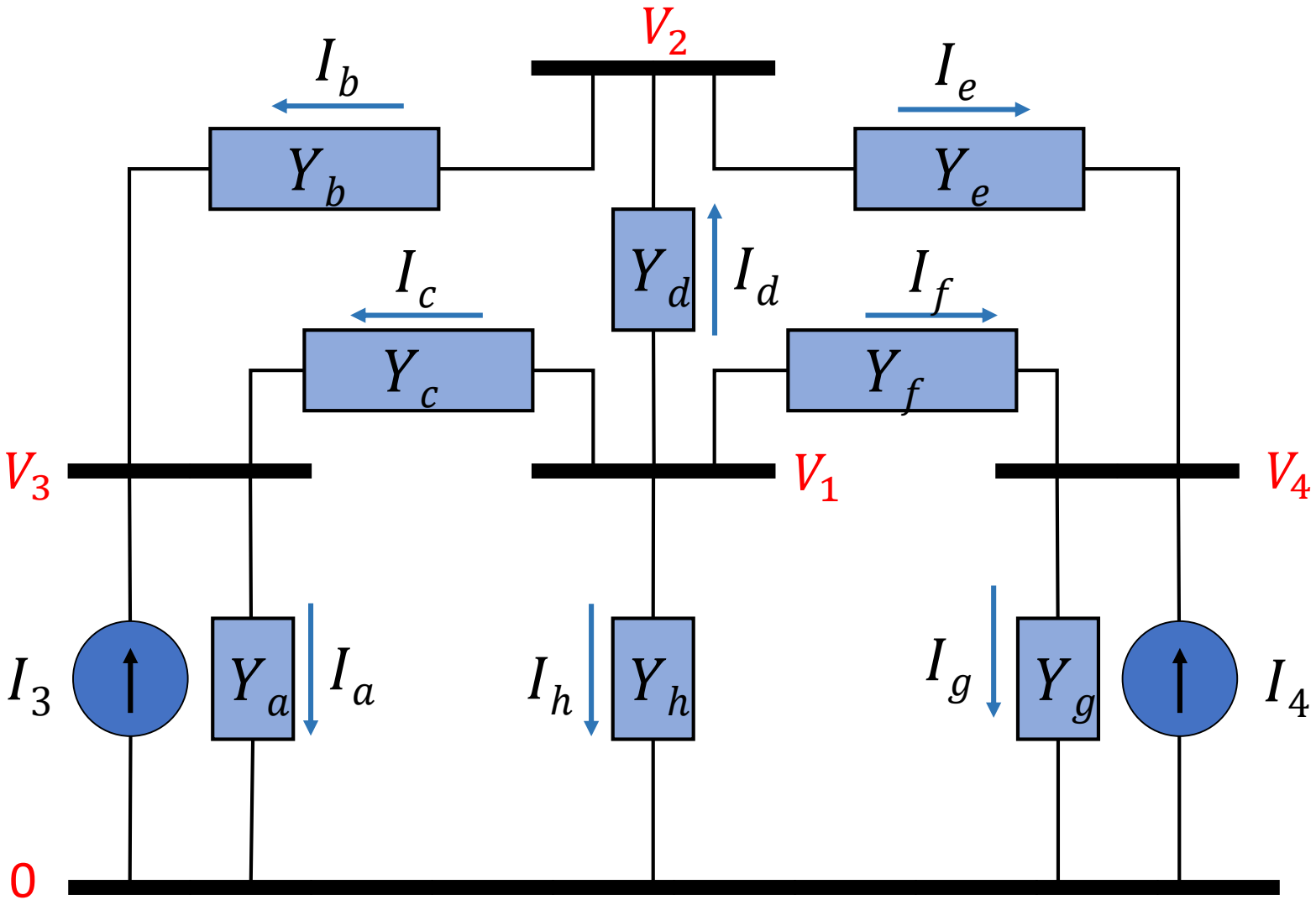
$$I_b - I_d + I_e = 0$$

$$I_a - I_b - I_c = I_3$$

$$I_g - I_e - I_f = I_4$$

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Step 5: Express branch currents in terms of node voltages



$$I_a = Y_a(V_3 - 0)$$

$$I_b = Y_b(V_2 - V_3)$$

$$I_c = Y_c(V_1 - V_3)$$

$$I_d = Y_d(V_1 - V_2)$$

$$I_e = Y_e(V_2 - V_4)$$

$$I_f = Y_f(V_1 - V_4)$$

$$I_g = Y_g(V_4 - 0)$$

$$I_h = Y_h(V_1 - 0)$$

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Step 5: Express KCL in terms of the node voltages

$$\begin{cases} I_c + I_d + I_f + I_h = 0 \\ I_b - I_d + I_e = 0 \\ I_a - I_b - I_c = I_3 \\ I_g - I_e - I_f = I_4 \end{cases}$$



$$\begin{cases} I_a = Y_a(V_3 - 0) \\ I_b = Y_b(V_2 - V_3) \\ I_c = Y_c(V_1 - V_3) \\ I_d = Y_d(V_1 - V_2) \\ I_e = Y_e(V_2 - V_4) \\ I_f = Y_f(V_1 - V_4) \\ I_g = Y_g(V_4 - 0) \\ I_h = Y_h(V_1 - 0) \end{cases}$$



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Step 5: Express KCL in terms of the node voltages

$$Y_c(V_1 - V_3) + Y_d(V_1 - V_2) + Y_f(V_1 - V_4) + Y_h(V_1 - 0) = 0$$
$$Y_b(V_2 - V_3) - Y_d(V_1 - V_2) + Y_e(V_2 - V_4) = 0$$
$$Y_a(V_3 - 0) - Y_b(V_2 - V_3) - Y_c(V_1 - V_3) = I_3$$
$$Y_g(V_4 - 0) - Y_e(V_2 - V_4) - Y_f(V_1 - V_4) = I_4$$

Group the terms according to the voltages

$$(Y_c + Y_d + Y_f + Y_h)V_1 - Y_dV_2 - Y_cV_3 - Y_fV_4 = 0$$
$$-Y_dV_1 + (Y_d + Y_b + Y_e)V_2 - Y_bV_3 - Y_eV_4 = 0$$
$$-Y_cV_1 - Y_bV_2 + (Y_a + Y_b + Y_c)V_3 = I_3$$
$$-Y_fV_1 - Y_eV_2 + (Y_g + Y_e + Y_f)V_4 = I_4$$

Step 6: Put these equations in matrix form

$$
\begin{cases}
(Y_c+Y_d + Y_f + Y_h)V_1 - Y_dV_2 - Y_cV_3 - Y_fV_4 = 0 \\
-Y_dV_1 + (Y_d + Y_b+Y_e)V_2 - Y_bV_3 - Y_eV_4 = 0 \\
-Y_cV_1 - Y_bV_2 + (Y_a + Y_b + Y_c)V_3 = I_3 \\
-Y_fV_1 - Y_eV_2 + (Y_g+Y_e + Y_f)V_4 = I_4
\end{cases}
$$

$$
\begin{pmatrix}
Y_c + Y_d + Y_f + Y_h & -Y_d & -Y_c & -Y_f \\
-Y_d & Y_d + Y_b + Y_e & -Y_b & -Y_e \\
-Y_c & -Y_b & Y_a + Y_b + Y_c & 0 \\
-Y_f & -Y_e & 0 & Y_g + Y_e + Y_f
\end{pmatrix}
\begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ I_3 \\ I_4 \end{pmatrix}
$$

Final form of the nodal voltage equations

$$\begin{pmatrix} Y_c + Y_d + Y_f + Y_h & -Y_d & -Y_c & -Y_f \\ -Y_d & Y_d + Y_b + Y_e & -Y_b & -Y_e \\ -Y_c & -Y_b & Y_a + Y_b + Y_c & 0 \\ -Y_f & -Y_e & 0 & Y_g + Y_e + Y_f \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ I_3 \\ I_4 \end{pmatrix}$$

Admittance matrix

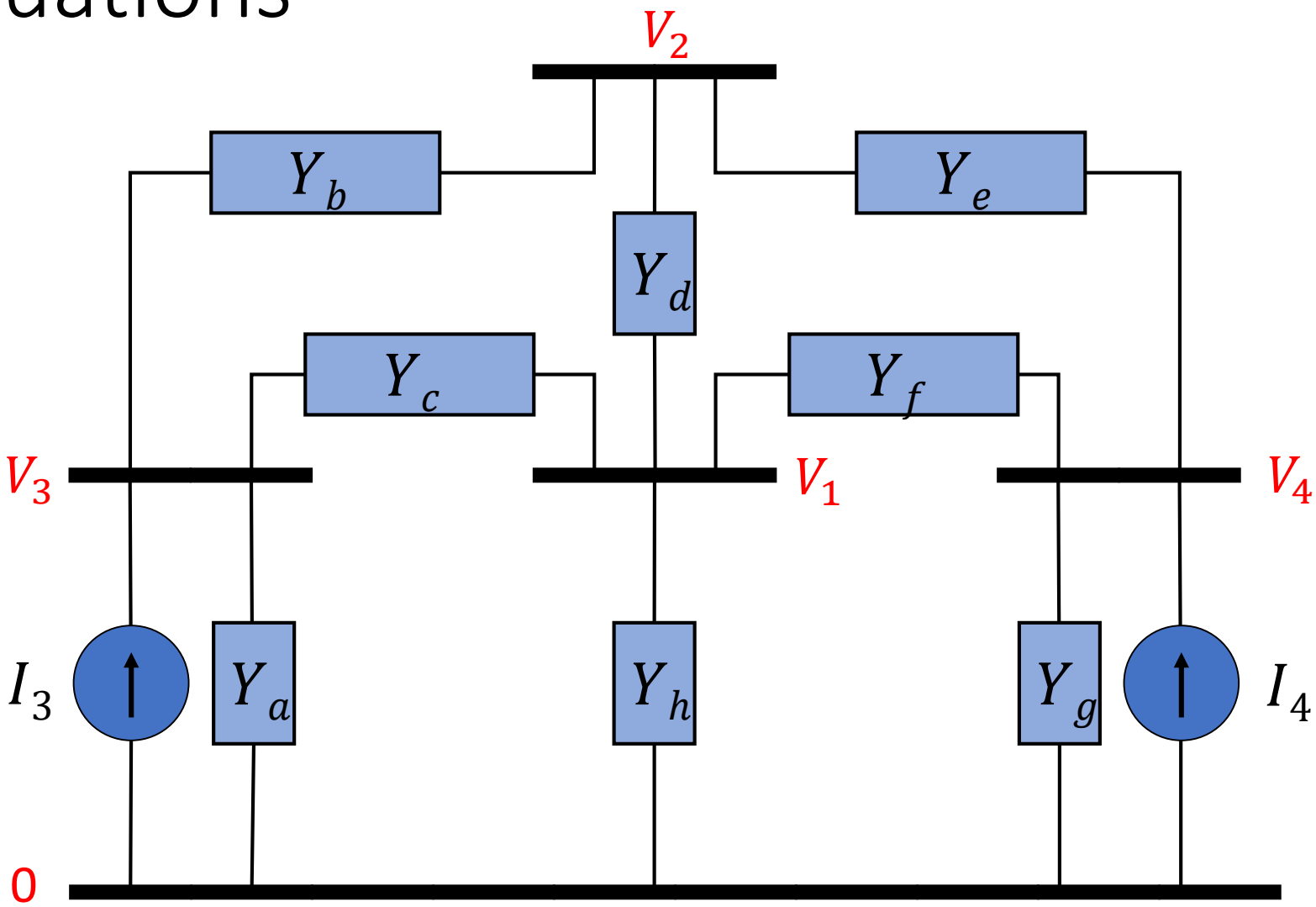
Unknown nodal voltages

Source currents

$$YV = I$$

Solving these equations

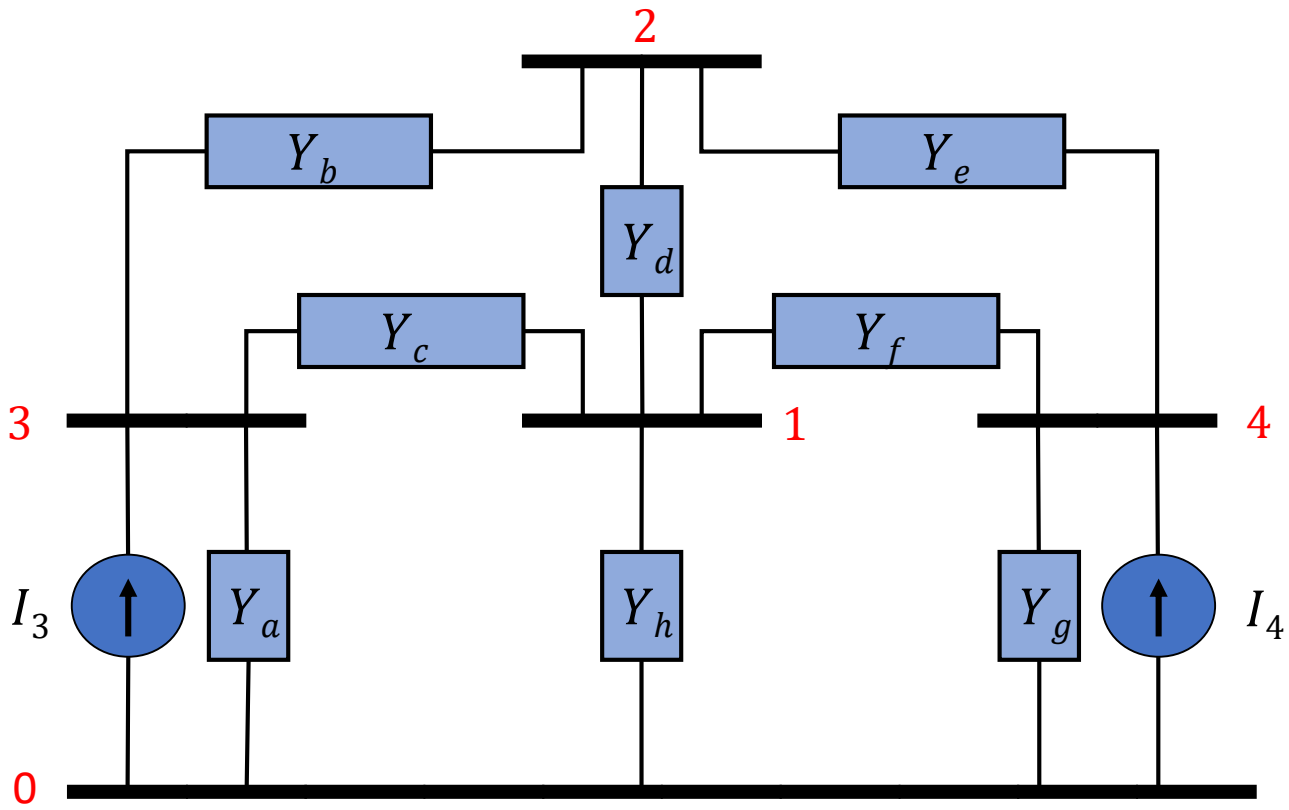
- System with N busses
- N equations
  - KCL at each bus
- N unknowns
  - Nodal voltages
- $YV = I$
- If Y^{-1} exists: $V = Y^{-1}I$



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$$\begin{pmatrix} Y_c + Y_d + Y_f + Y_h & -Y_d & -Y_c & -Y_f \\ -Y_d & Y_d + Y_b + Y_e & -Y_b & -Y_e \\ -Y_c & -Y_b & Y_a + Y_b + Y_c & 0 \\ -Y_f & -Y_e & 0 & Y_g + Y_e + Y_f \end{pmatrix}$$

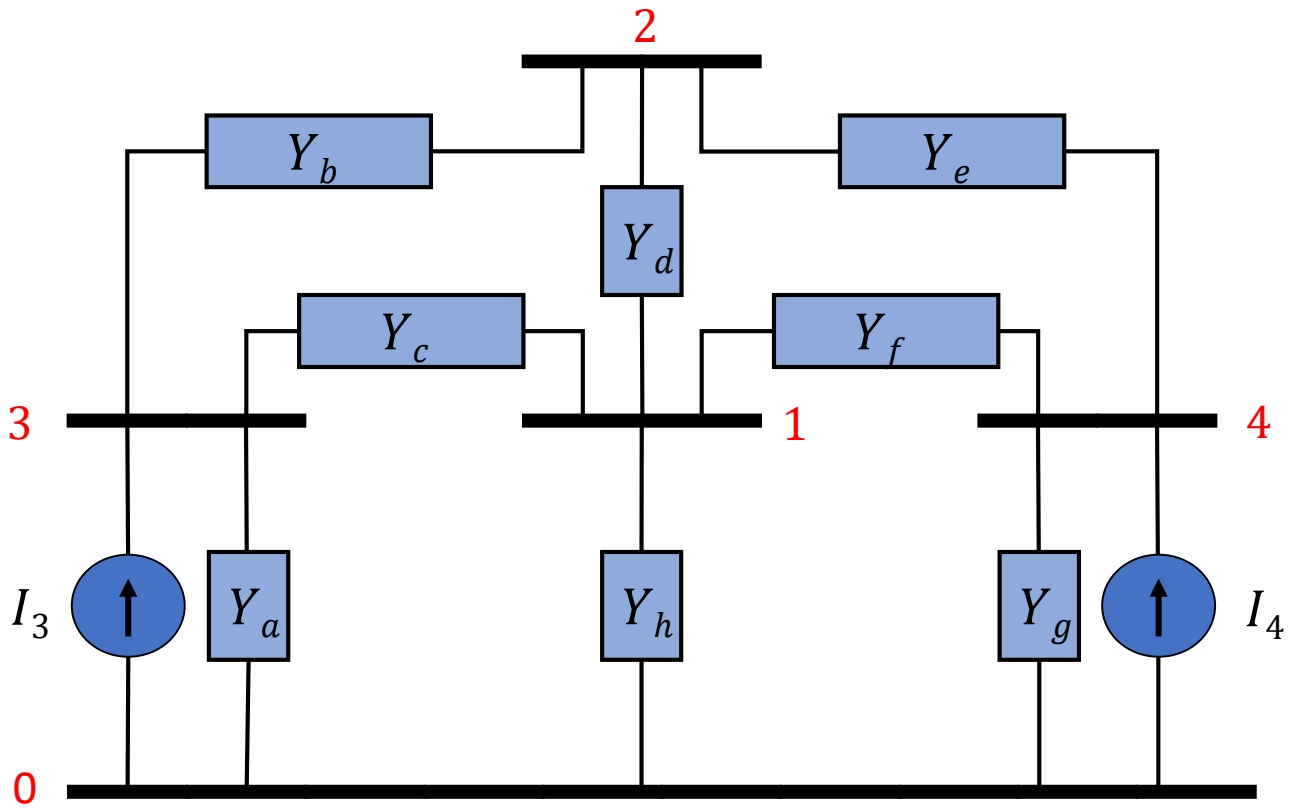
What is the structure of the admittance matrix?



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$$\begin{pmatrix} Y_c + Y_d + Y_f + Y_h & -Y_d & -Y_c & -Y_f \\ -Y_d & Y_d + Y_b + Y_e & -Y_b & -Y_e \\ -Y_c & -Y_b & Y_a + Y_b + Y_c & 0 \\ -Y_f & -Y_e & 0 & Y_g + Y_e + Y_f \end{pmatrix}$$

- Rows and columns of Y correspond to nodes of the network
- Diagonal element (i, i):
  - Sum of the admittances connected to node i
- Off-diagonal element (i, j):
  - Minus admittance between nodes i and j
- Given the network data, we can build the admittance matrix directly



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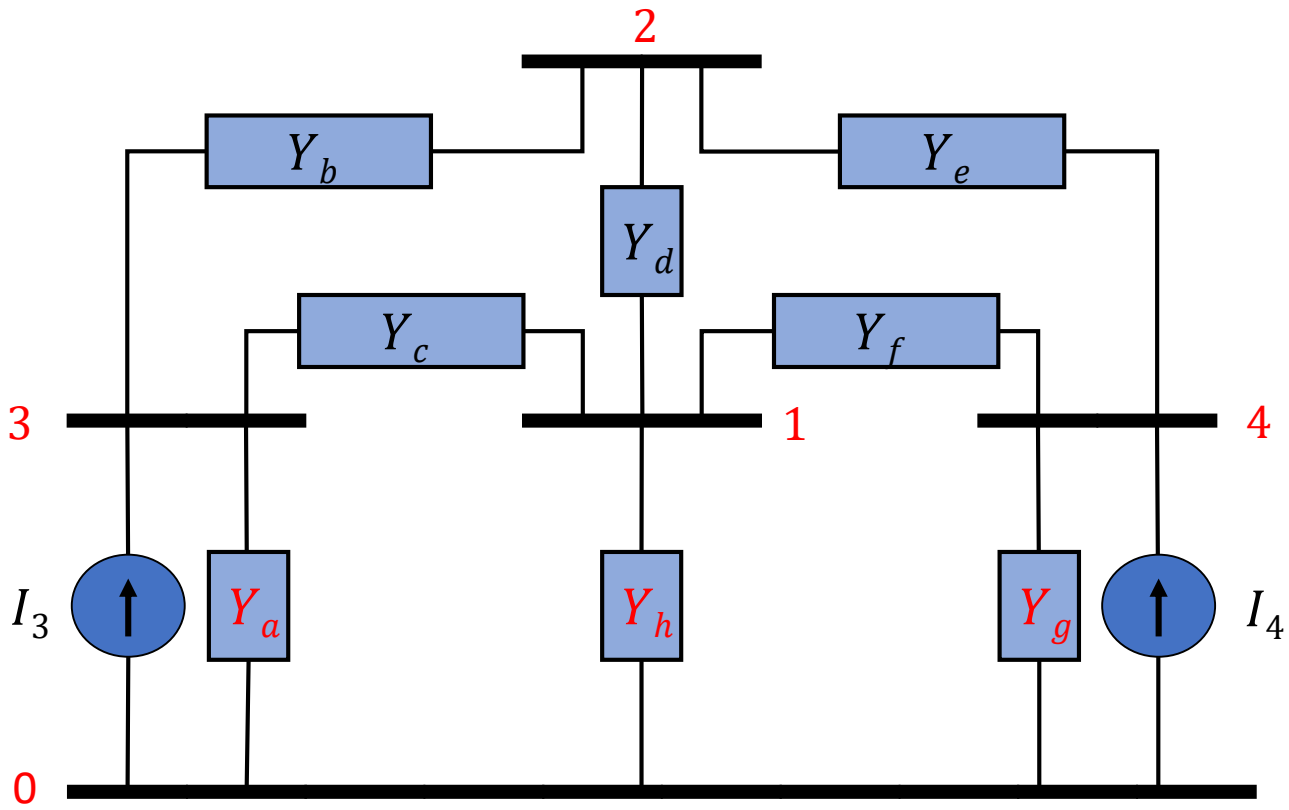
$$\begin{pmatrix} Y_c + Y_d + Y_f + Y_h & -Y_d & -Y_c & -Y_f \\ -Y_d & Y_d + Y_b + Y_e & -Y_b & -Y_e \\ -Y_c & -Y_b & Y_a + Y_b + Y_c & 0 \\ -Y_f & -Y_e & 0 & Y_g + Y_e + Y_f \end{pmatrix}$$

- The diagonal elements are never zero
  - There is always at least one branch connected to each node
- The admittances to ground only appear in the diagonal elements

→ A linear combination of rows and columns will never add to zero

→The admittance matrix is never singular

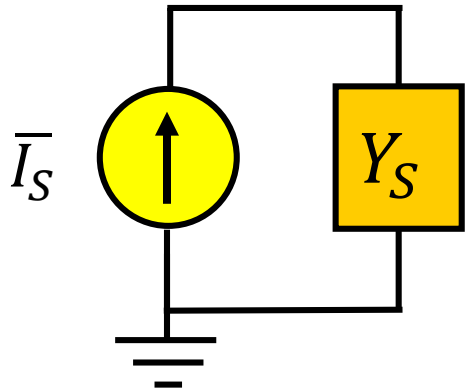
→ Y^{-1} always exists



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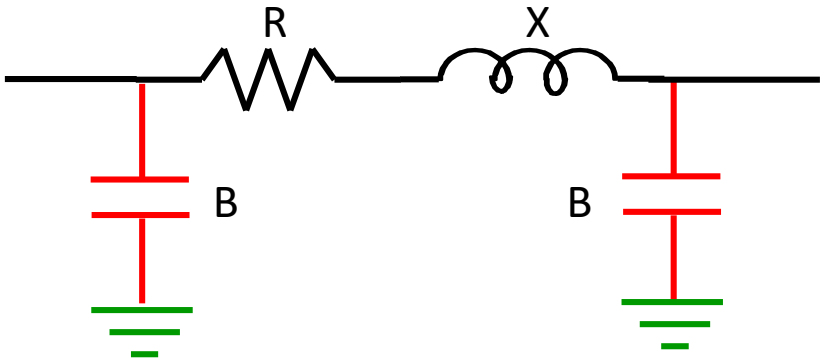
Admittances between nodes and ground

- Parallel admittances of sources



- Loads

- Shunt admittances of lines



Terms of the admittance matrix

- DC networks
  - Branches have a real-valued resistance

  → Branches have real-valued admittances

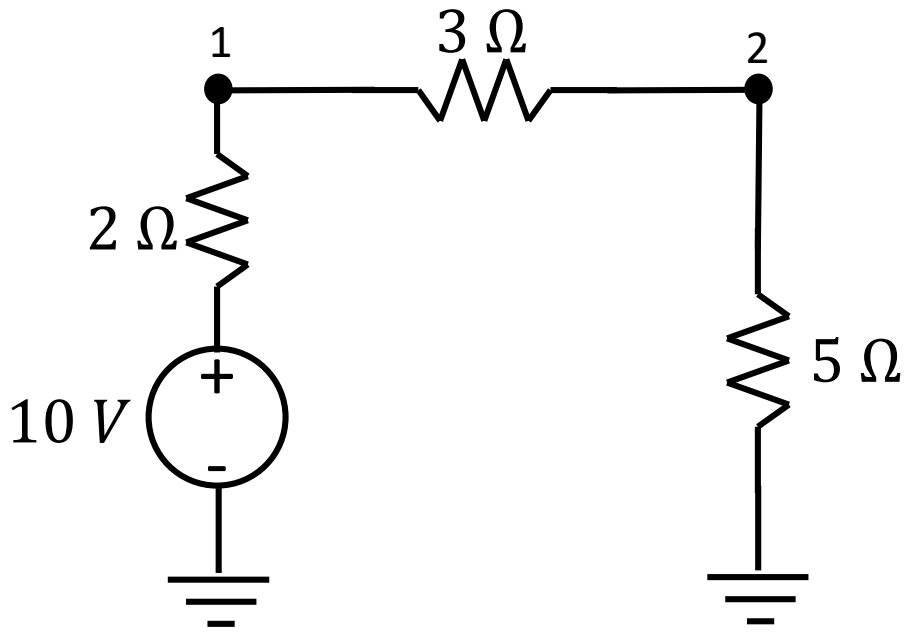
  → All the terms of the admittance matrix are real
- AC networks
  - Branches have complex-valued impedances

  → Branches have complex-valued admittances

  → All the terms of Y are complex
- Inverting a complex matrix is just as feasible as inverting a real matrix
  - Particularly with a computer!

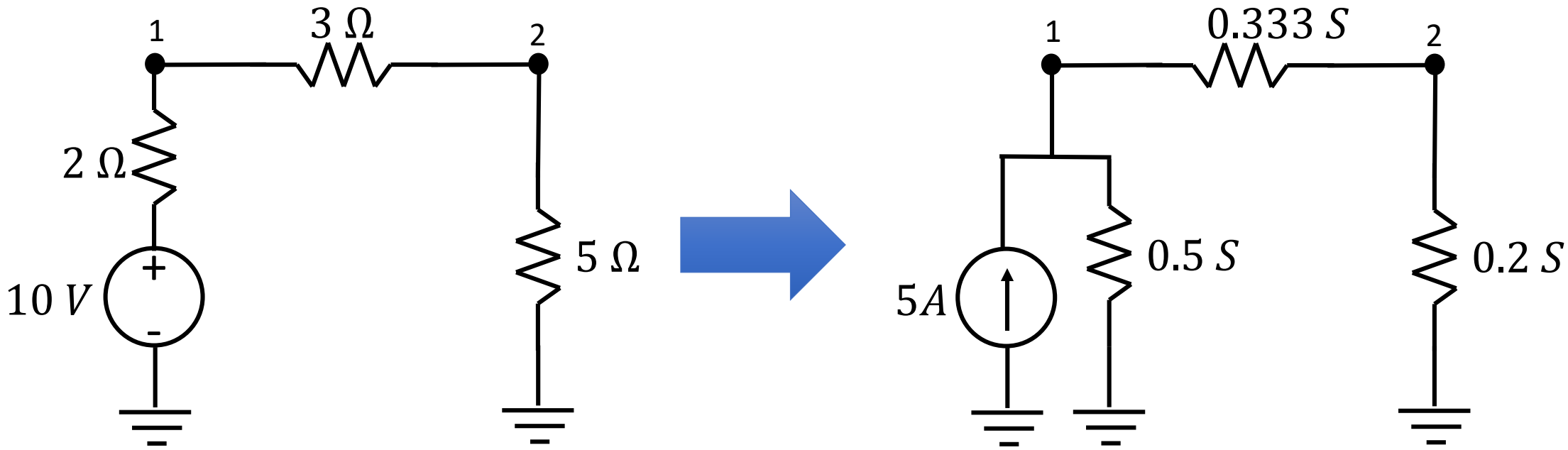
Example 8.1

Build the nodal equations of the DC network shown below.



Example 8.1

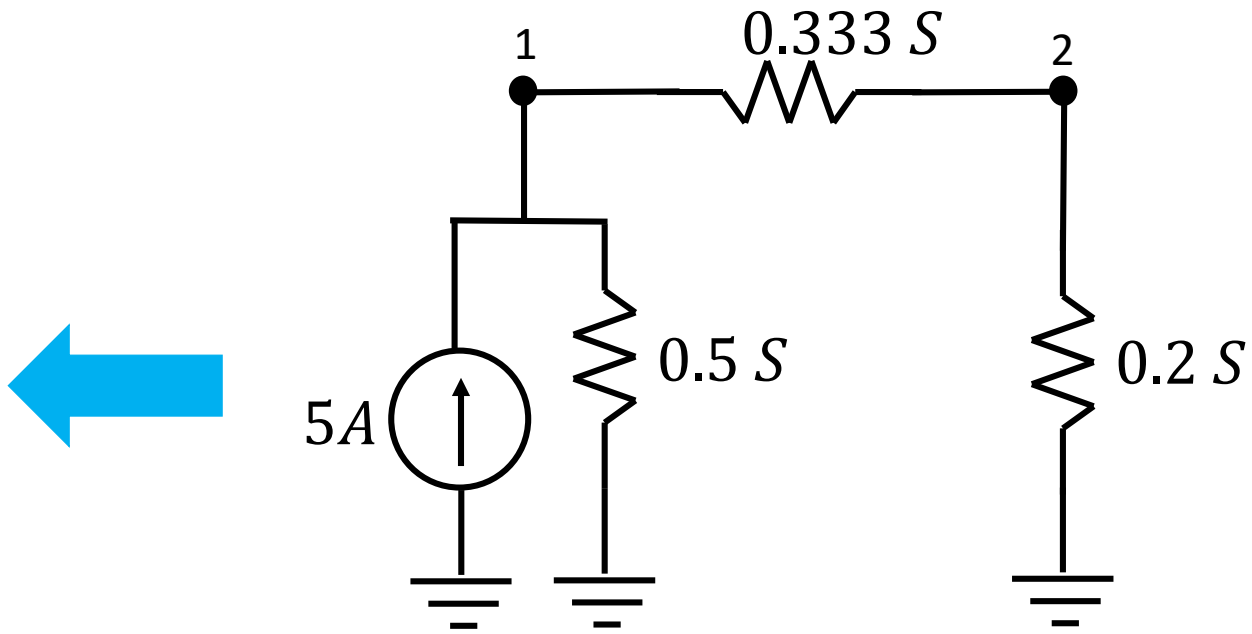
Convert voltage sources to current sources and impedances to admittances



Example 8.1

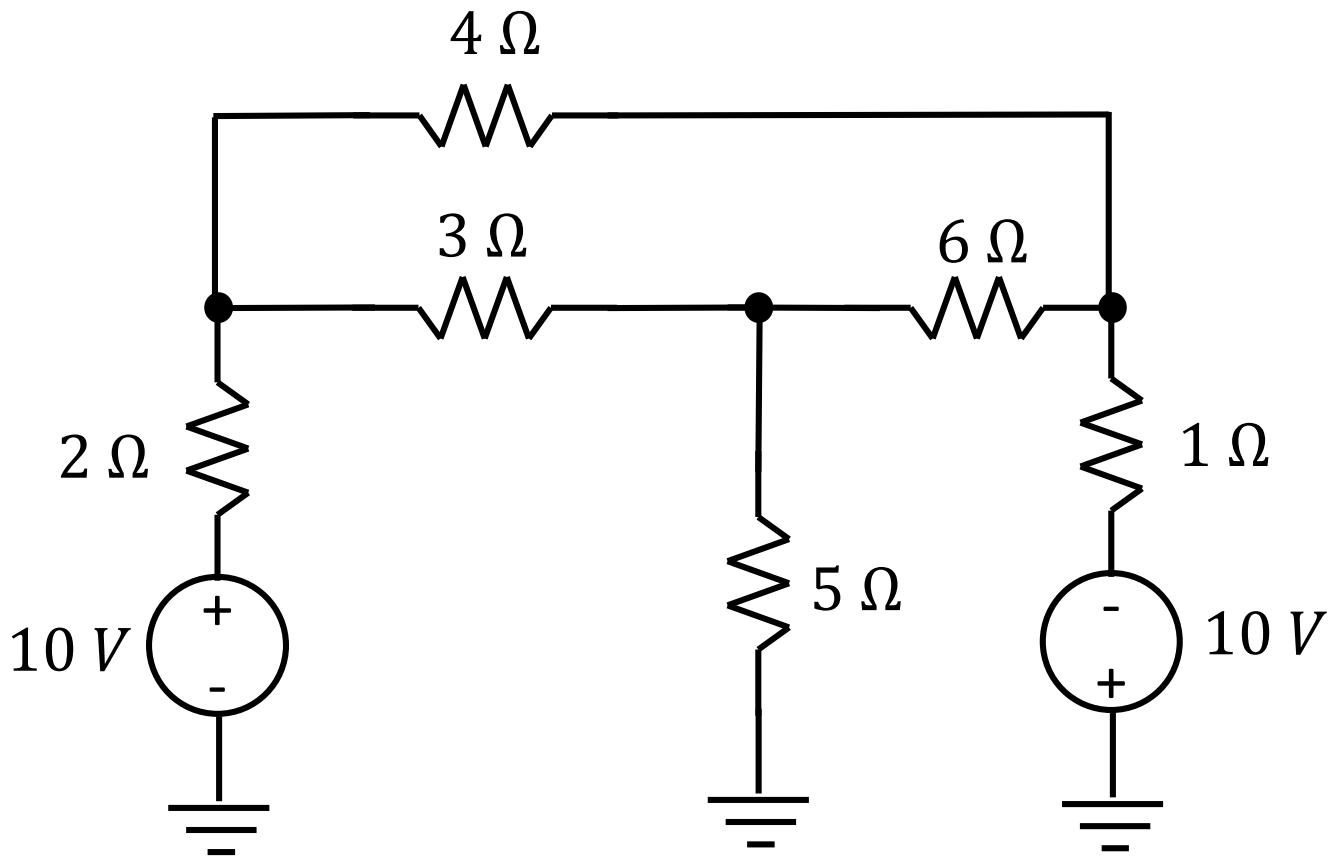
Nodal equations

$$\begin{pmatrix} 0.833 & -0.333 \\ -0.333 & 0.533 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$



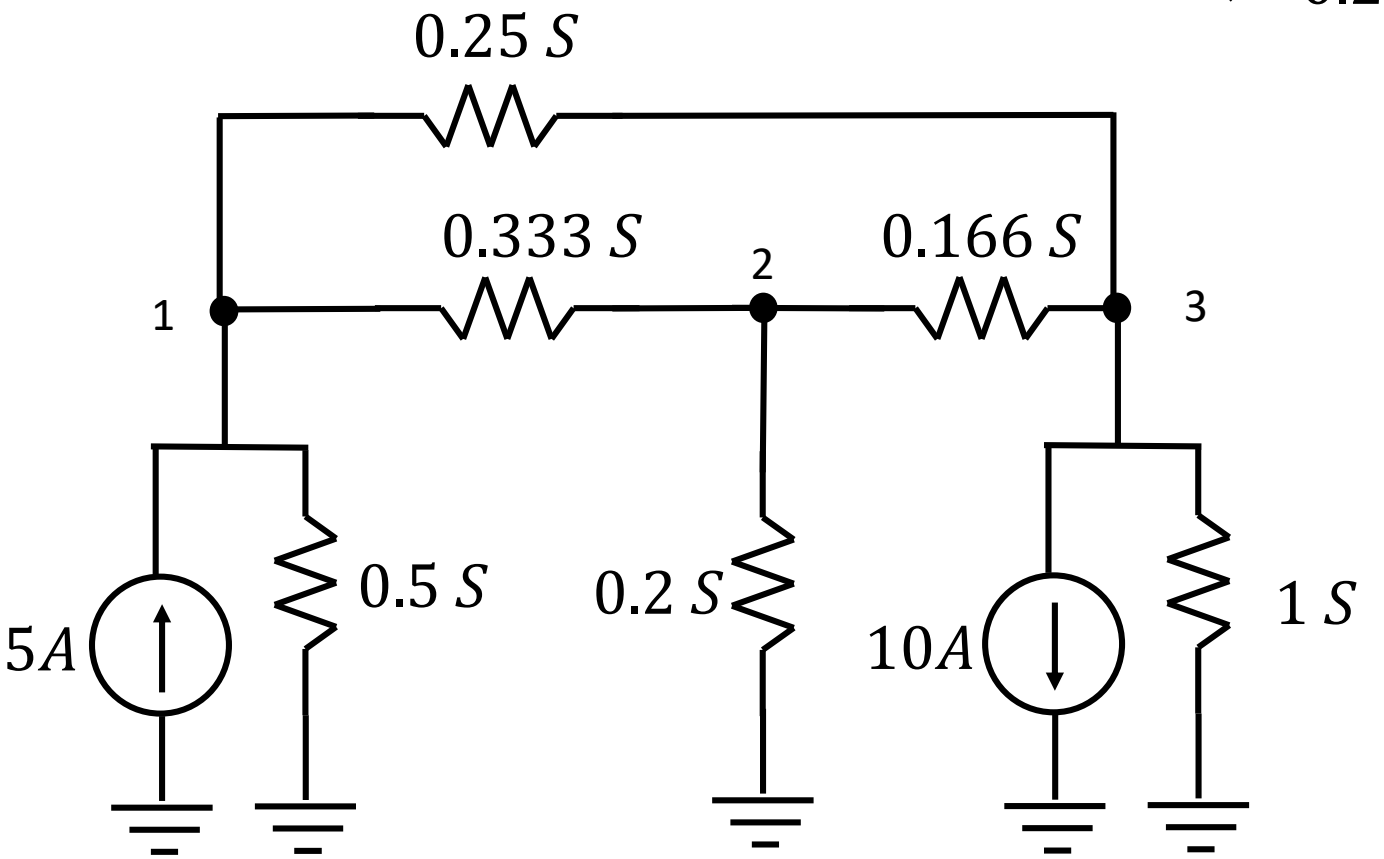
Example 8.2

Build the nodal equations of the DC network shown below.



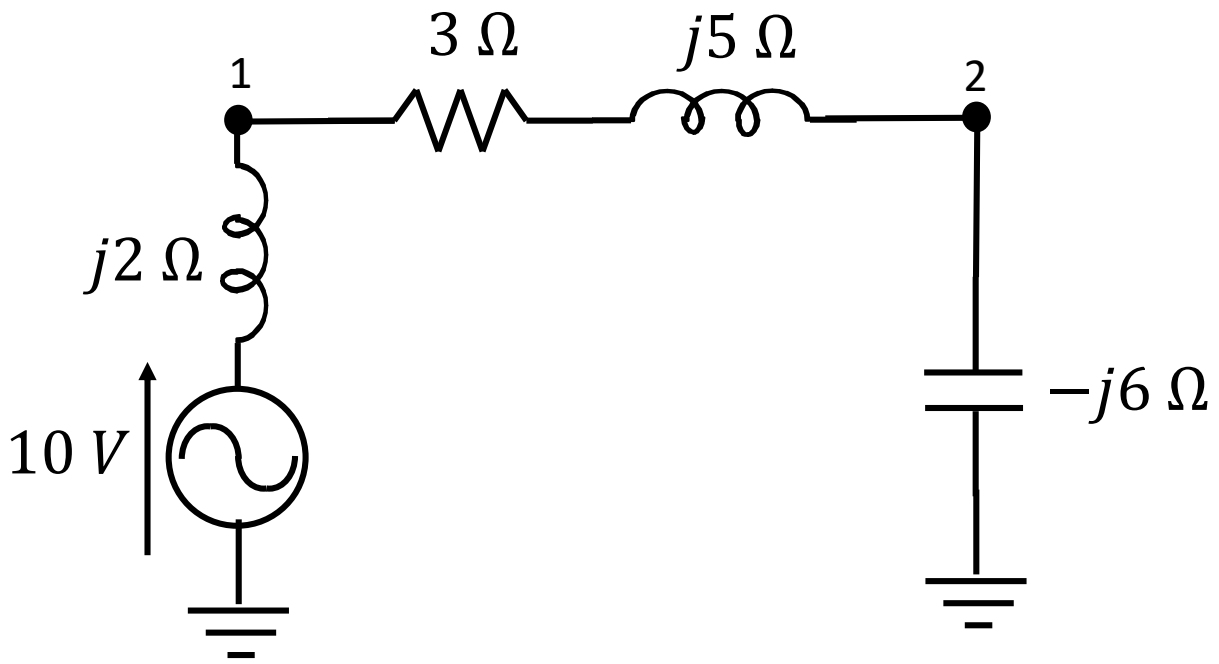
Example 8.2

$$\begin{pmatrix} 1.083 & -0.333 & -0.25 \\ -0.333 & 0.7 & -0.166 \\ -0.25 & -0.166 & 1.416 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -10 \end{pmatrix}$$



Example 8.3

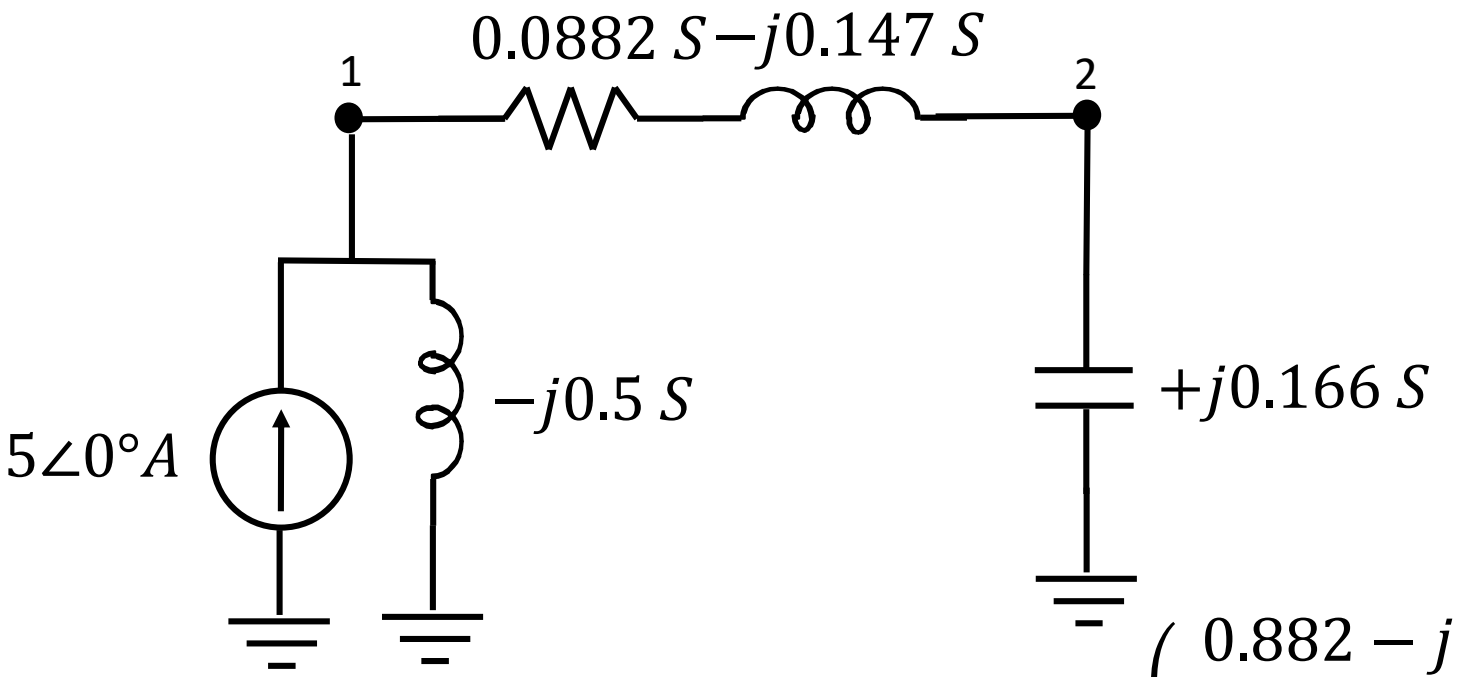
Build the nodal equations of the AC network shown below.



Example 8.3

Convert voltage source to current source and impedances to admittances

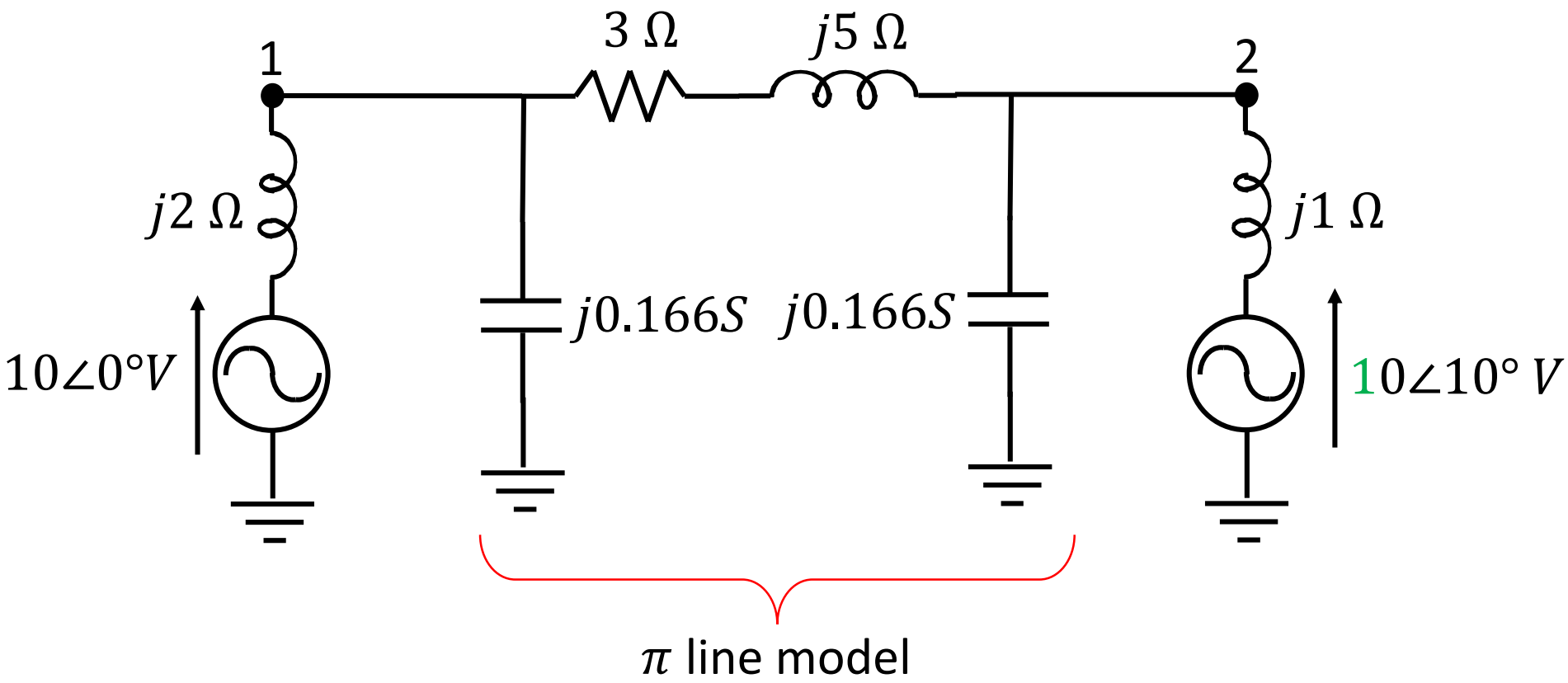
$$z = 3 + j5\ \Omega \rightarrow y = \frac{1}{3+j5} = \frac{3-j5}{3^2+5^2} = 0.882 - j0.147$$



$$\begin{pmatrix} 0.882 - j0.647 & -0.882 + j0.147 \\ -0.882 + j0.147 & 0.882 + j0.019 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

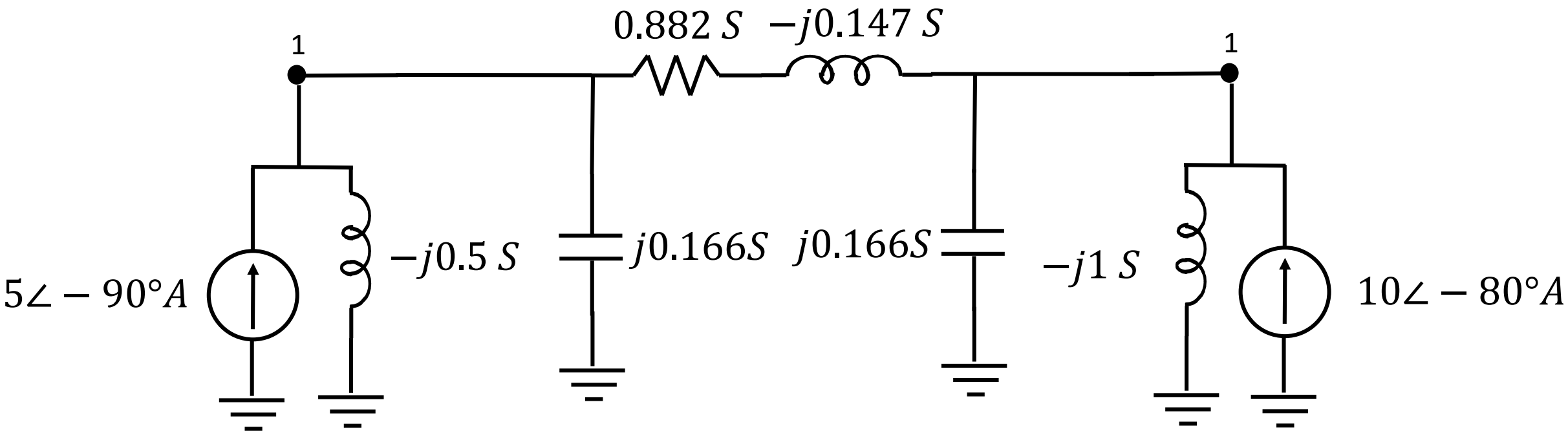
Example 8.4

Build the nodal equations of the AC network shown below.



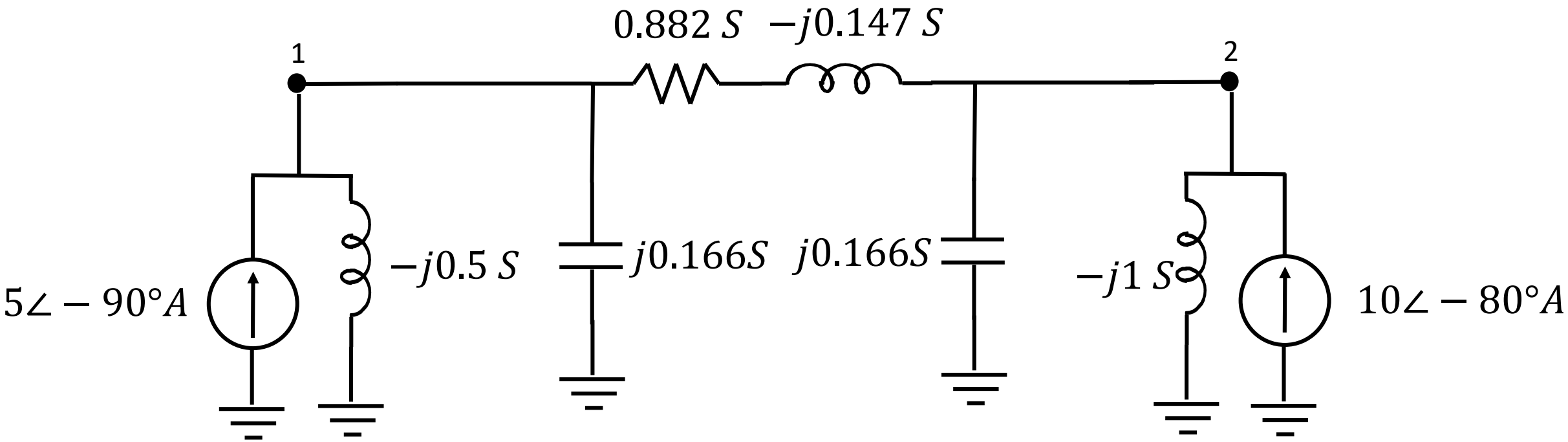
Example 8.4

Convert voltage source to current source and impedances to admittances

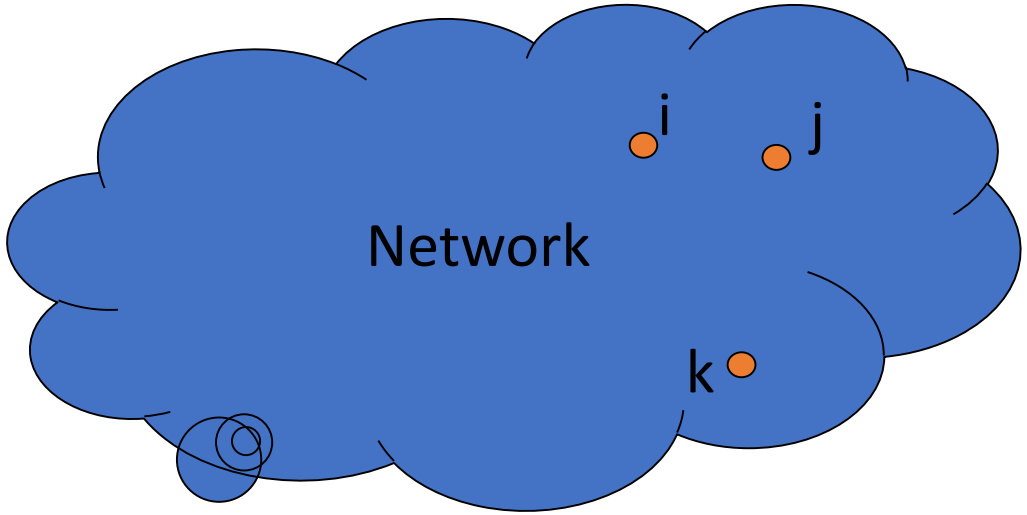


Example 8.4

$$\begin{pmatrix} 0.882 - j0.481 & -0.882 + j0.147 \\ -0.882 + j0.147 & 0.882 - j0.981 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 5\angle -90° \\ 10\angle -80° \end{pmatrix}$$



Admittance matrix of a modified network

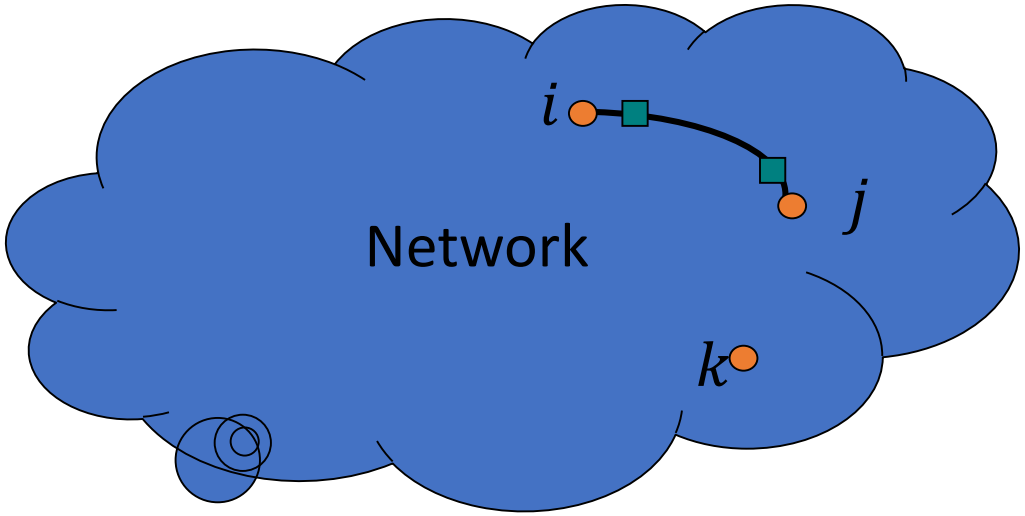


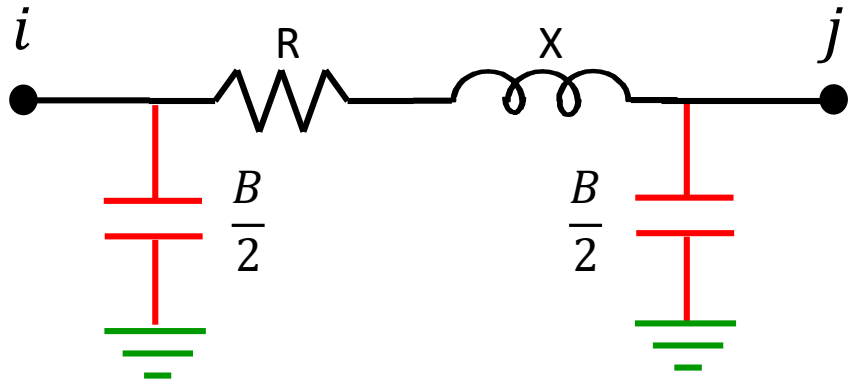
$$YV = I$$

- Opening and closing switches changes the topology of the network
- How do the equations change when:
  - A line is connected between nodes *i* and *j* ?
  - A line is disconnected between nodes *i* and *j* ?
  - A source is connected at node *k* ?

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Adding a line between nodes *i* and *j*





$z = R + jX$ is the series impedance of the line

$y = \frac{1}{z} = \frac{1}{R+jX}$ is the series admittance of the line

$\frac{B}{2}$ is the shunt admittance at each end of the line

$$Y_{ii} = Y_{ii} + \mathrm{y} + \frac{B}{2}$$

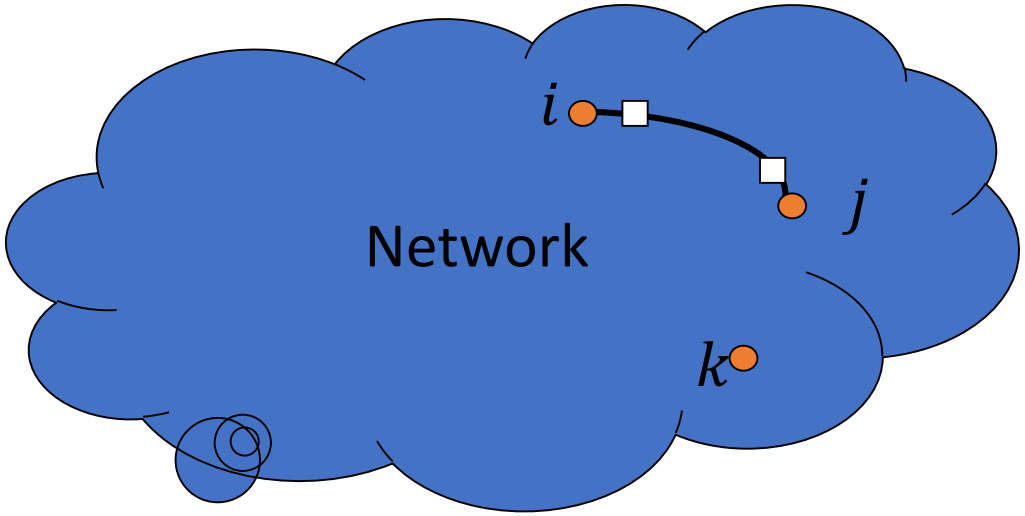
$$Y_{jj} = Y_{jj} + \mathrm{y} + \frac{B}{2}$$

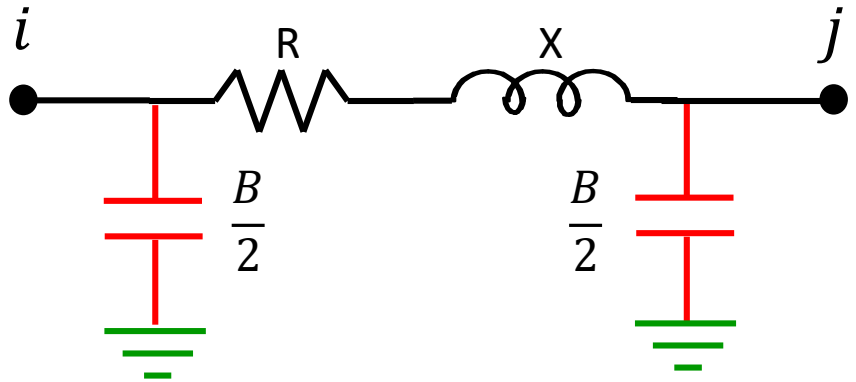
$$Y_{ij} = Y_{ij} - \mathrm{y}$$

$$Y_{ji} = Y_{ji} - \mathrm{y}$$

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Removing a line between nodes *i* and *j*





$z = R + jX$ is the series impedance of the line

$y = \frac{1}{z} = \frac{1}{R+jX}$ is the series admittance of the line

$\frac{B}{2}$ is the shunt admittance at each end of the line

$$Y_{ii} = Y_{ii} - \mathrm{y} - \frac{B}{2}$$

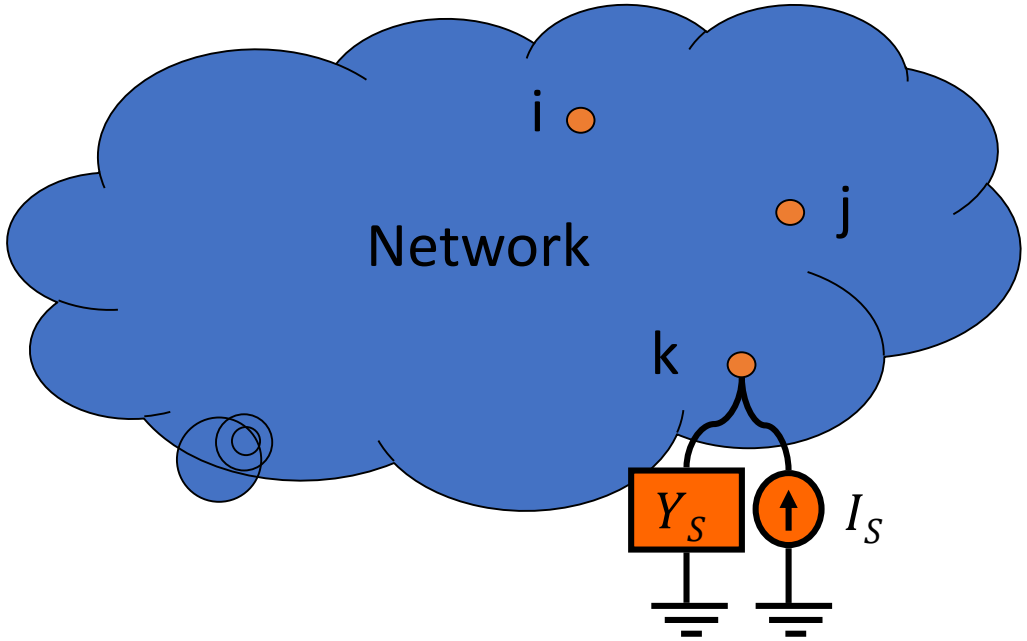
$$Y_{jj} = Y_{jj} - \mathrm{y} - \frac{B}{2}$$

$$Y_{ij} = Y_{ij} + \mathrm{y}$$

$$Y_{ji} = Y_{ji} + \mathrm{y}$$

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Adding a source at node *k*



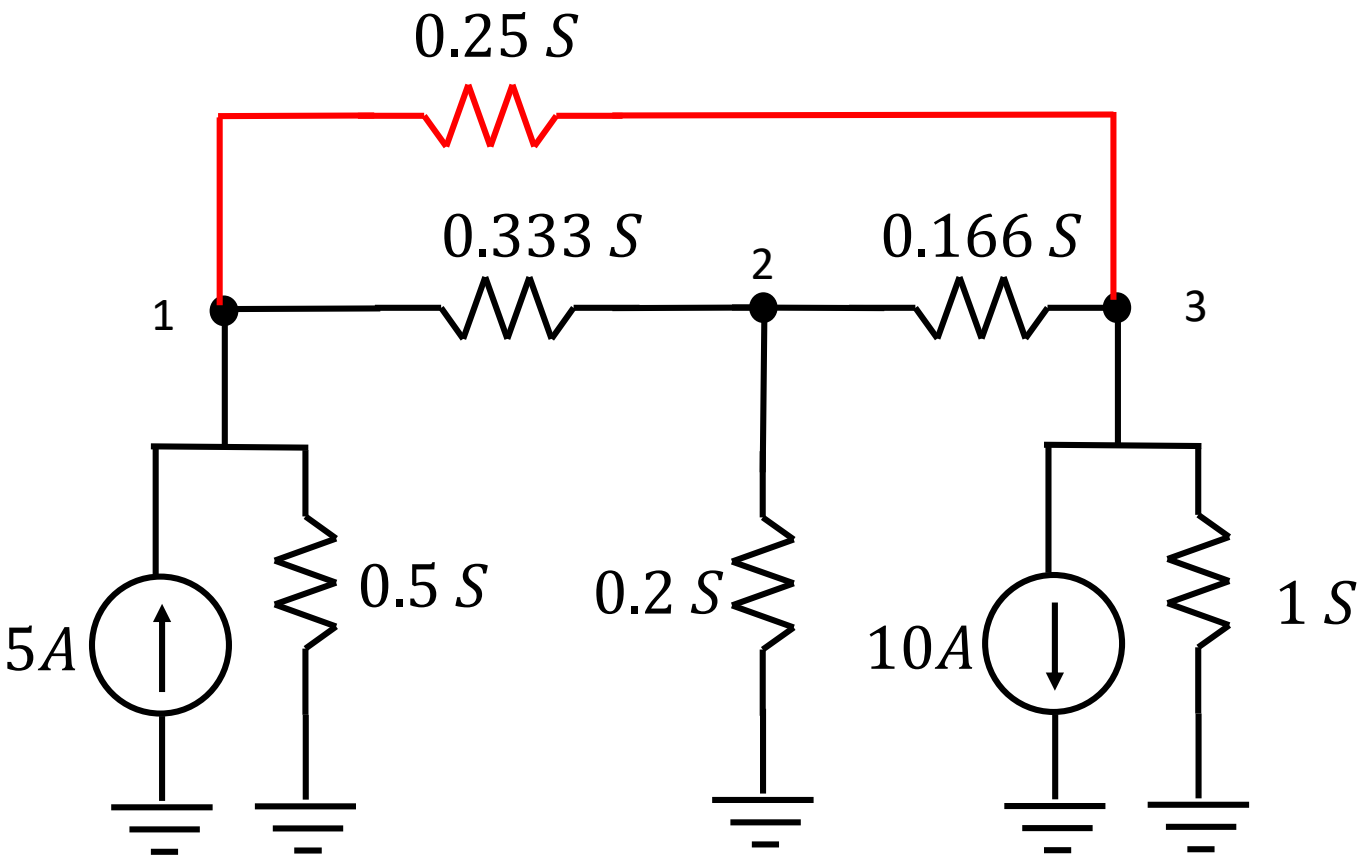
$$Y_{kk} \rightarrow Y_{kk} + Y_S$$

$$I_k \rightarrow I_k + I_S$$

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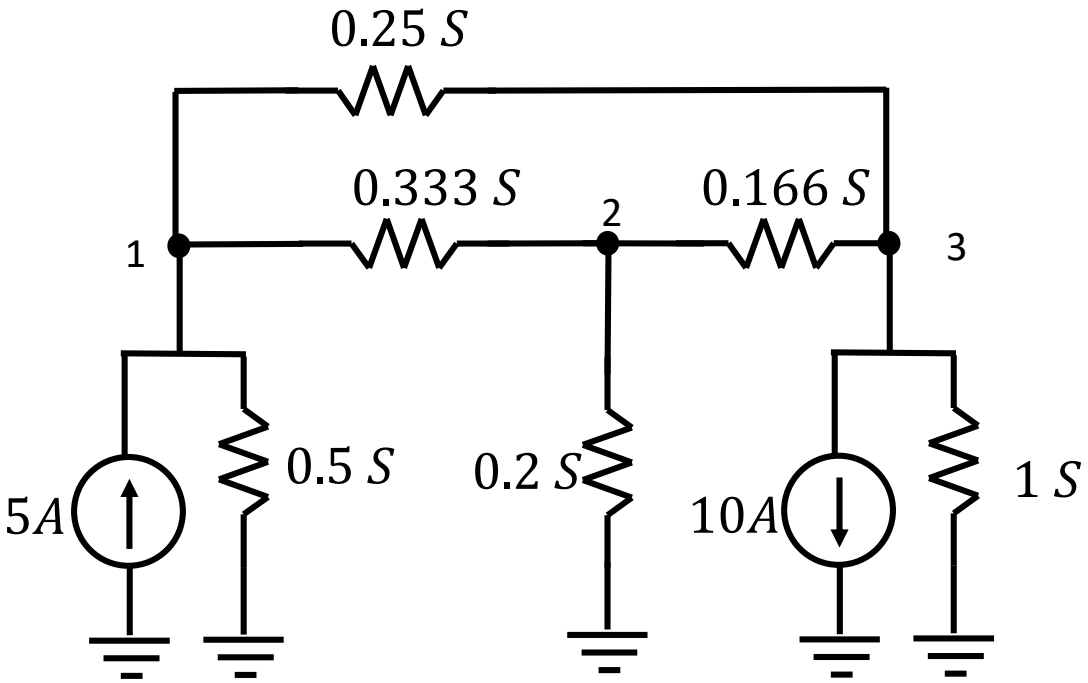
Example 8.5

Modify the admittance matrix built in Example 8.2 to reflect the removal of the admittance between nodes 1 and 3

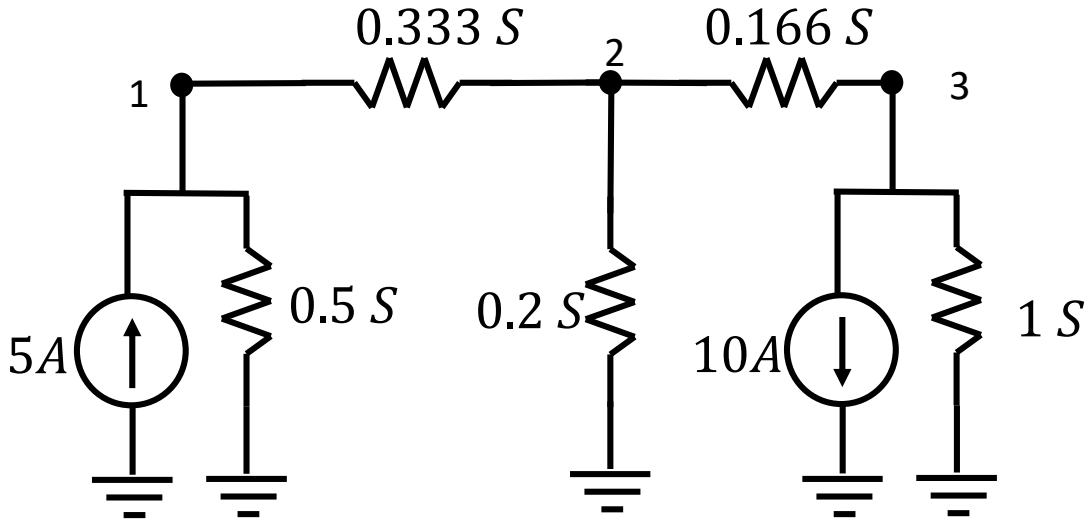


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Example 8.5



$$\begin{pmatrix} 1.083 & -0.333 & -0.25 \\ -0.333 & 0.7 & -0.166 \\ -0.25 & -0.166 & 1.416 \end{pmatrix}$$

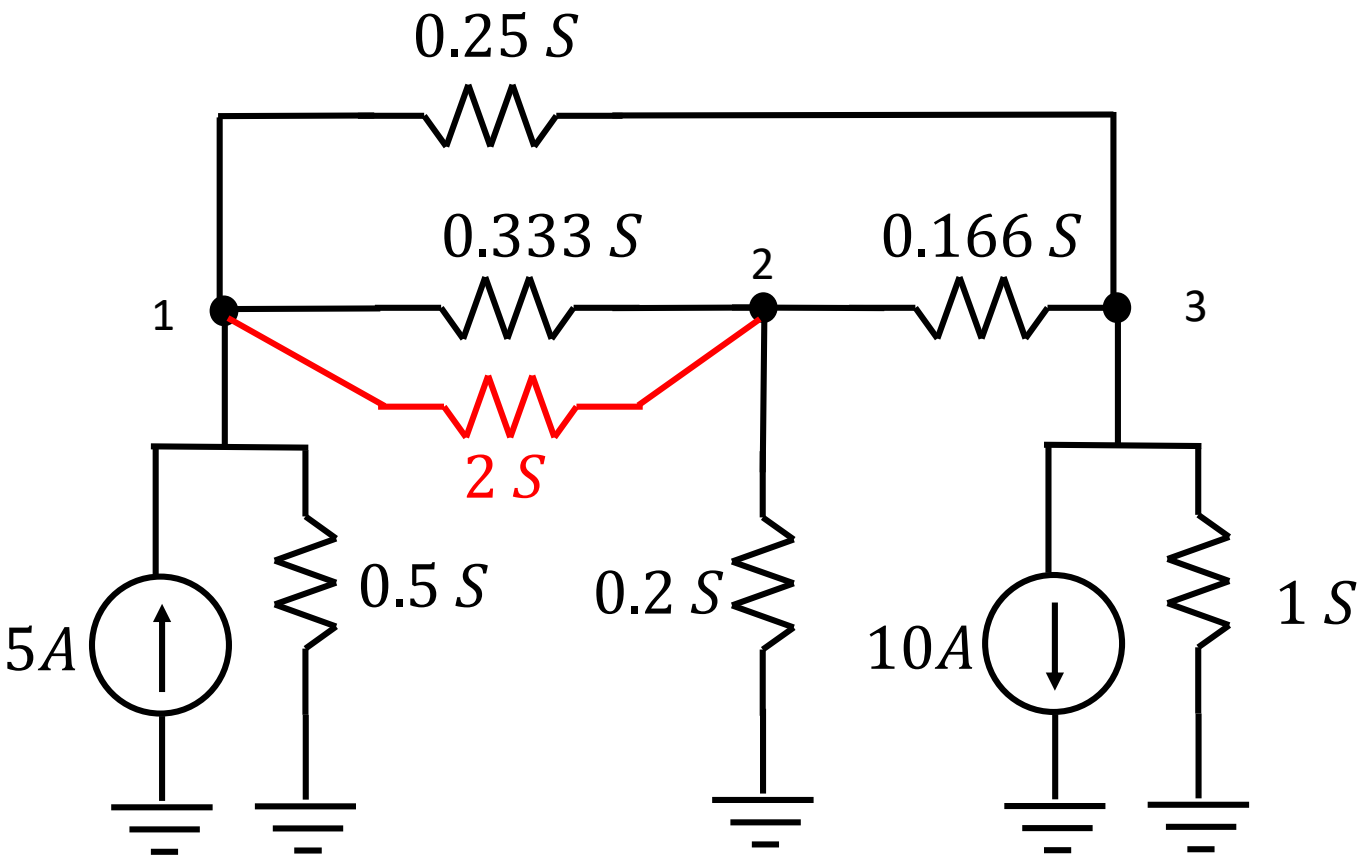


$$\begin{pmatrix} 0.833 & -0.333 & 0 \\ -0.333 & 0.7 & -0.166 \\ 0 & -0.166 & 1.116 \end{pmatrix}$$

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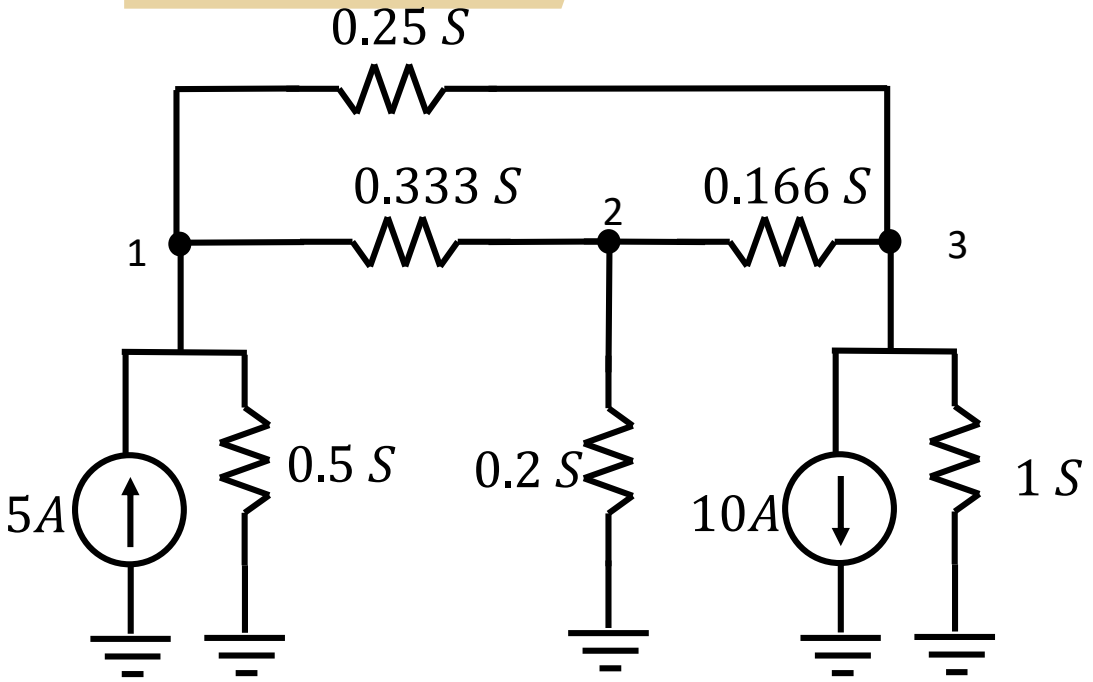
Example 8.6

Modify the admittance matrix built in Example 8.2 to reflect the addition of a 2S admittance between nodes 1 and 2

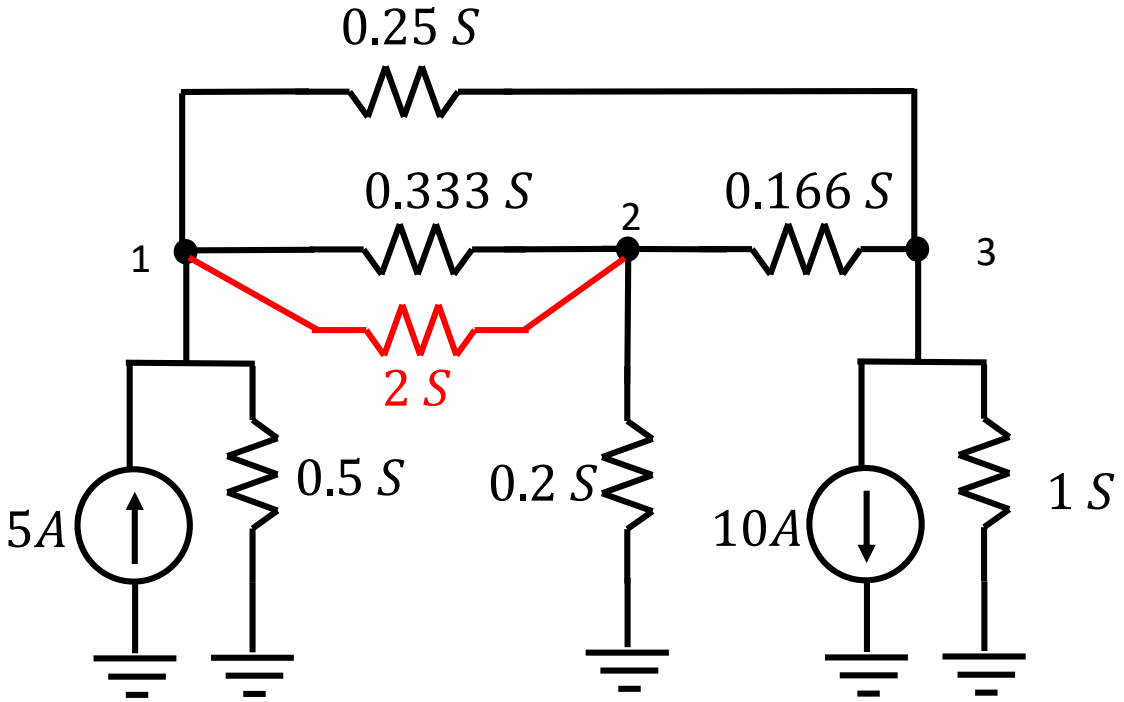


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Example 8.6



$$\begin{pmatrix} 1.083 & -0.333 & -0.25 \\ -0.333 & 0.7 & -0.166 \\ -0.25 & -0.166 & 1.416 \end{pmatrix}$$



$$\begin{pmatrix} 3.083 & -2.333 & -0.25 \\ -2.333 & 2.7 & -0.166 \\ -0.25 & -0.166 & 1.416 \end{pmatrix}$$

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Solving the nodal equations

- We have a systematic method for forming the nodal equations

$$YV = I$$

- Systematic = something that a computer can handle
- Need a systematic method for solving these equations
- Three steps:
  - LU factorization
  - Forward solution
  - Backward solution

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LU Factorization

$$\begin{pmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{pmatrix} = \begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix} \cdot \begin{pmatrix} 1 & U_{12} & U_{13} & U_{14} \\ 0 & 1 & U_{23} & U_{24} \\ 0 & 0 & 1 & U_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Any non-singular matrix can be decomposed into the product of a lower triangular matrix and an upper triangular matrix

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How do we decompose *A* into *A*= *LU*?

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} . \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

Proceed by row

Processing each element determines the unknown shown in red

$$a_{11} = l_{11} \qquad a_{21} = l_{21} \qquad a_{31} = l_{31}$$

$$a_{12} = l_{11}u_{12} \qquad a_{22} = l_{21}u_{12} + l_{22} \qquad a_{32} = l_{31}u_{12} + l_{32}$$

$$a_{13} = l_{11}u_{13} \qquad a_{23} = l_{21}u_{13} + l_{22}u_{23} \qquad a_{33} = l_{31}u_{13} + l_{32}u_{23} + l_{33}$$

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Forward and Backward Solutions

$$YV = I$$

$$LUV = I$$

Let $UV = V'$

$$\begin{pmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix} \begin{pmatrix} V_1' \\ V_2' \\ V_3' \\ V_4' \end{pmatrix} = \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix}$$

Solve $LV' = I$ for V'

Solve $UV = V'$ for V

$$\begin{pmatrix} 1 & U_{12} & U_{13} & U_{14} \\ 0 & 1 & U_{23} & U_{24} \\ 0 & 0 & 1 & U_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} = \begin{pmatrix} V_1' \\ V_2' \\ V_3' \\ V_4' \end{pmatrix}$$

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Example (1)

$$\begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{pmatrix} \cdot \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}$$

Assume that we have factorized the matrix Y

$$\begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{pmatrix} = \begin{pmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \cdot \begin{pmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, we must solve:

$$\begin{pmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \cdot \begin{pmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}$$

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Example (2)

$$\begin{pmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} V_1' \\ V_2' \\ V_3' \end{pmatrix}$$

The V' are new unknowns
They are related to the original unknowns V through these equations

Replacing in the complete equations, we get:

$$\begin{pmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \cdot \begin{pmatrix} V_1' \\ V_2' \\ V_3' \end{pmatrix} = \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}$$

This is a set of equations in V'
It is easy to solve because it is triangular

$$L_{11}V_1' = I_1 \quad \Rightarrow \quad V_1' = I_1 / L_{11}$$

$$L_{21}V_1' + L_{22}V_2' = I_2 \quad \Rightarrow \quad V_2' = \left(I_2 - L_{21}V_1'\right) / L_{22}$$

$$L_{31}V_1' + L_{32}V_2' + L_{33}V_3' = I_3 \quad \Rightarrow \quad V_3' = \left(I_3 - L_{31}V_1' - L_{32}V_2'\right) / L_{33}$$

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Example (3)

The forward solution of $LV' = I$ has given us the values of V'.
We can now solve the following system of equations for V:

$$\begin{pmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} V_1' \\ V_2' \\ V_3' \end{pmatrix}$$

The solution proceeds from the bottom (backwards) because the matrix is upper triangular.

$$V_3 = V_3'$$

$$V_2 = V_2' - U_{23} V_3$$

$$V_1 = V_1' - U_{12} V_2 - U_{13} V_3$$

Done!

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Advantage of LU factorization

- If $YV = I$ needs to be solved for multiple right hand side vectors I, the factorization does not need to be repeated
- Since the factorization is the most time-consuming part of the process, this makes repeat solutions very efficient

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Sparsity of the admittance matrix

- Consider a network with 1500 buses
- Size of Y: 1,500 x 1,500
- Number of elements: 2,250,000
- Number of non-zero elements:
  - On average each bus has three neighbors
  - 4 per row on average (diagonal + 3 neighbors) 4 x 1500 = 6,000 non-zeros
  - Proportion of zero elements: 99.7%
- Y is a sparse matrix
- Y is still non-singular (invertible)
- Solving $YV = I$ is much faster if we take advantage of the fact that most of the terms are zero

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Sparsity Programming

- Take advantage of the sparsity of the admittance matrix Y
  - Store only the non-zero elements
  - Carry out only the multiplications that involve non-zero elements
  - Reduce the amount of memory needed
  - Reduce the computing time required by not handling the zeros
- All commercial power system analysis programs make use of sparsity programming

→ They can solve networks with tens of thousands of nodes in seconds

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Mathematical formulation of the power flow problem

Nodal power balance

- Conventional nodal analysis:

$$I = YV$$

- Expresses Kirchhoff's Current Law or balance of current at each node

- In power systems, we work with powers rather than currents

- Formulate nodal analysis in terms of the balances of active and reactive powers at each node/bus

Nodal power balance

$$I = YV$$

Row k of this matrix equation expresses KCL at node k

Current injected at node k

$$\overline{I_k} = \sum_{i=1}^{N} Y_{ki}\overline{V_i}$$

Sum of the currents flowing in the branches connected to node k

N: number of nodes in the network

$$\overline{I_k}^* = \sum_{i=1}^{N} Y_{ki}^*\overline{V_i^*}$$

Complex power injected at node k

$$\overline{S_k} = \overline{V_k}\overline{I_k}^* = \sum_{i=1}^{N} Y_{ki}^*\overline{V_k}\overline{V_i}^*$$

Sum of the complex powers flowing in the branches connected to node k

A bit of complex algebra…

$$\overline{S_k} = \sum_{i=1}^{N} Y_{ki}^{*} \overline{V_k} \overline{V_i}^{*}$$

$Y_{ki} = G_{ki} + jB_{ki}$: element (k, i) of the admittance matrix

$\overline{V_k} = V_k \angle \theta_k$: voltage at node k

$\overline{V_i} = V_i \angle \theta_i$: voltage at node i

$$\overline{S_k} = \sum_{i=1}^{N} (G_{ki} - jB_{ki}) V_k \angle \theta_k V_i \angle -\theta_i$$

$$\overline{S_k} = \sum_{i=1}^{N} (G_{ki} - jB_{ki}) V_k V_i \angle (\theta_k - \theta_i)$$

$$\overline{S_k} = \sum_{i=1}^{N} (G_{ki} - jB_{ki}) V_k V_i [cos(\theta_k - \theta_i) + jsin(\theta_k - \theta_i)]$$

More complex algebra…

$$\overline{S_k} = \sum_{i=1}^{N} (G_{ki} - jB_{ki}) V_k V_i [cos(\theta_k - \theta_i) + jsin(\theta_k - \theta_i)]$$

Let $\theta_{ki} = (\theta_k - \theta_i)$ = difference in voltage phase angle between nodes k and i

$$\overline{S_k} = \sum_{i=1}^{N} (G_{ki} - jB_{ki}) V_k V_i [cos\theta_{ki} + jsin\theta_{ki}]$$

$$\overline{S_k} = \sum_{i=1}^{N} V_k V_i \{[G_{ki} cos\theta_{ki} + B_{ki} sin\theta_{ki}] + j[G_{ki} sin\theta_{ki} - B_{ki} cos\theta_{ki}]\}$$

Finally…

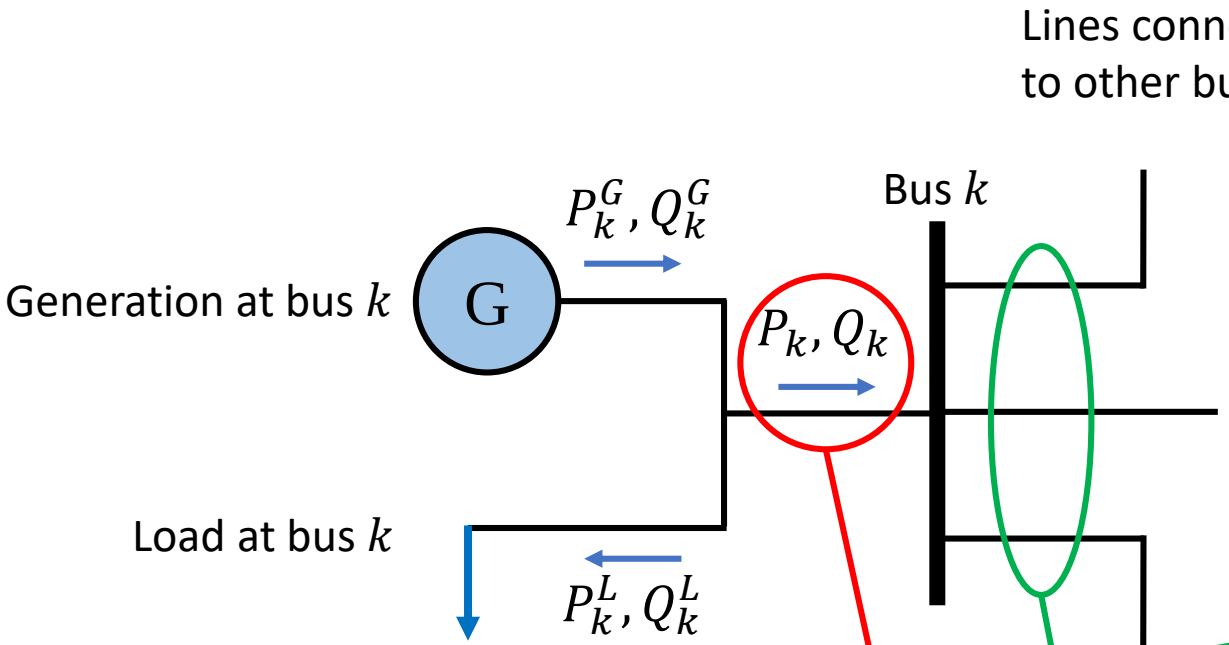
$$\overline{S_k} = \sum_{i=1}^{N} V_k V_i \{[G_{ki} cos\theta_{ki} + B_{ki} sin\theta_{ki}] + j[G_{ki} sin\theta_{ki} - B_{ki} cos\theta_{ki}]\}$$

$\overline{S_k} = P_k + jQ_k$: active and reactive powers injected at bus k

$$\begin{cases} P_k = \sum_{i=1}^{N} V_k V_i [G_{ki} cos\theta_{ki} + B_{ki} sin\theta_{ki}] \\ Q_k = \sum_{i=1}^{N} V_k V_i [G_{ki} sin\theta_{ki} - B_{ki} cos\theta_{ki}] \end{cases}$$

Power flow equations

Physical interpretation of the power flow equations



Active power injection at bus k: $P_k = P_k^G - P_k^L$

Reactive power injection at bus k: $Q_k = Q_k^G - Q_k^L$

$$P_k = \sum_{i=1}^{N} V_k V_i [G_{ki} cos\theta_{ki} + B_{ki} sin\theta_{ki}]$$

$$Q_k = \sum_{i=1}^{N} V_k V_i [G_{ki} sin\theta_{ki} - B_{ki} cos\theta_{ki}]$$

The power flow equations

$$\begin{cases} P_k = \sum_{i=1}^{N} V_k V_i [G_{ki} cos\theta_{ki} + B_{ki} sin\theta_{ki}] \\ Q_k = \sum_{i=1}^{N} V_k V_i [G_{ki} sin\theta_{ki} - B_{ki} cos\theta_{ki}] \end{cases}$$

P_k: Active power injection at bus k

Q_k: Reactive power injection at bus k

N: Number of buses in the network

V_k: Magnitude of the voltage at node k

V_i: Magnitude of the voltage at node i

θ_k: Phase of the voltage at node k

θ_i: Phase of the voltage at node i

$\theta_{ki} = \theta_k - \theta_i$

G_{ki}: Real part of (i, k) term of matrix Y

B_{ki}: Imaginary part of (i, k) term of matrix Y

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The power flow equations

$$P_k = \sum_{i=1}^{N} V_k V_i [G_{ki} cos\theta_{ki} + B_{ki} sin\theta_{ki}]$$

$$Q_k = \sum_{i=1}^{N} V_k V_i [G_{ki} sin\theta_{ki} - B_{ki} cos\theta_{ki}]$$

N buses → 2N equations

4N variables: P_k, Q_k, θ_k, V_k at each node k

Must have 2N unknown variables and 2N known variables

At each bus whether P_k, Q_k, θ_k, or V_k is known or unknown depends on the type of bus

Three bus types:

- PQ bus
- PV bus
- Slack bus

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PQ bus

$$\begin{cases} P_k = \sum_{i=1}^{N} V_k V_i [G_{ki} cos\theta_{ki} + B_{ki} sin\theta_{ki}] \\ Q_k = \sum_{i=1}^{N} V_k V_i [G_{ki} sin\theta_{ki} - B_{ki} cos\theta_{ki}] \end{cases}$$

P_k, Q_k are given and thus known

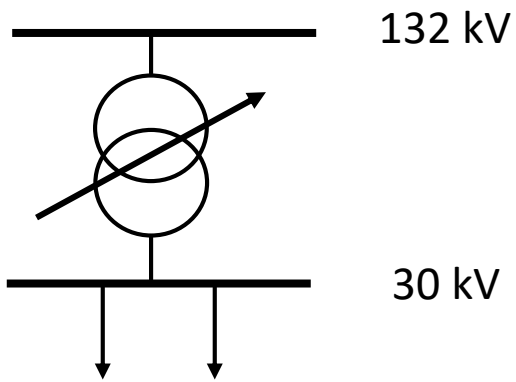
θ_k, V_k are unknown

Buses with only a load are PQ buses

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Why model loads as constant P&Q?

Voltage regulator adjusts the taps to keep the voltage on the low voltage side constant as the transmission level voltage changes. The consumers (and the loads) thus see a constant voltage. P&Q thus remain constant.



Note: more sophisticated load models are used in dynamic calculations

PV bus

$$P_k = \sum_{i=1}^{N} V_k V_i [G_{ki} cos\theta_{ki} + B_{ki} sin\theta_{ki}]$$

$$Q_k = \sum_{i=1}^{N} V_k V_i [G_{ki} sin\theta_{ki} - B_{ki} cos\theta_{ki}]$$

A generator typically:

- Produces a known amount of active power
- Maintains the voltage at its terminals constant

If there is a generator connected at bus k:

P_k and V_k are given and thus known

θ_k and Q_k are unknown

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Slack bus – A bit of history

- Under normal conditions, generation must be adjusted to match the load and keep the frequency constant at its nominal value
- Secondary frequency control distributes these changes among all the generators
- Before the invention of secondary frequency control, in each region all generators except one operated according to a fixed schedule
- The remaining generator adjusted its output to keep the frequency constant using primary frequency control
- In other words, this generator picked up the slack
- The bus where it was located was the slack bus

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Why do we need a slack bus for the power flow?

- Goal of the power flow:
  - Given the powers injected by the generator and extracted by the loads, calculate the voltages and flows
- Power flow models the steady state and thus assumes that the system is in balance and the frequency constant

$$\underbrace{\sum_{g \in G} P_g}_{\text{Known}} = \underbrace{\sum_{l \in L} P_l}_{\text{Known}} + \underbrace{losses\ in\ the\ network}_{\text{Unknown because depends on the flows!}}$$

CONTRADICTION

→ Need to provide some slack generation

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Slack bus

$$P_k = \sum_{i=1}^{N} V_k V_i [G_{ki} cos\theta_{ki} + B_{ki} sin\theta_{ki}]$$

$$Q_k = \sum_{i=1}^{N} V_k V_i [G_{ki} sin\theta_{ki} - B_{ki} cos\theta_{ki}]$$

- Choose a generator bus as the slack bus
- Leave P_k unspecified
- Since this is a generator bus, specify V_k
- Leave Q_k unspecified
- Need one more specified variable
- Need a reference for the angles
- Set $\theta_k = 0°$ at the slack bus

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Numbers of equations and variables

- N buses
- 2 N equations (2 at each bus)
- 4 N variables (P, Q, V, θ at each bus)

- 2N known variables (2 at each bus):
  - P, Q at PQ buses
  - P, V at PV buses
  - V, θ at the slack bus
- 2N unknown variables (2 at each bus):
  - V, θ at PQ buses
  - Q, θ at PV buses
  - P, Q at the slack bus

Example

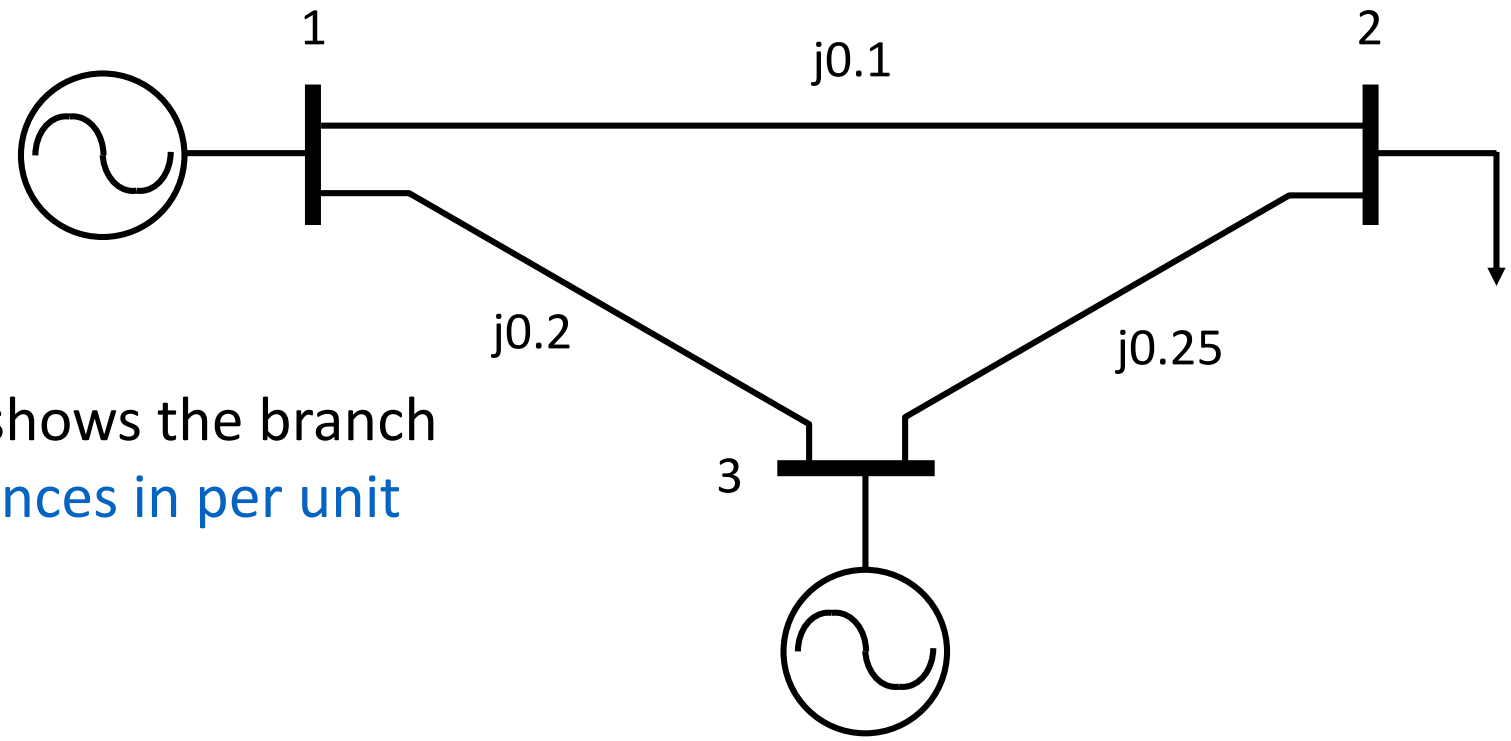


Figure shows the branch impedances in per unit

Write the power flow equations for this system

Y matrix

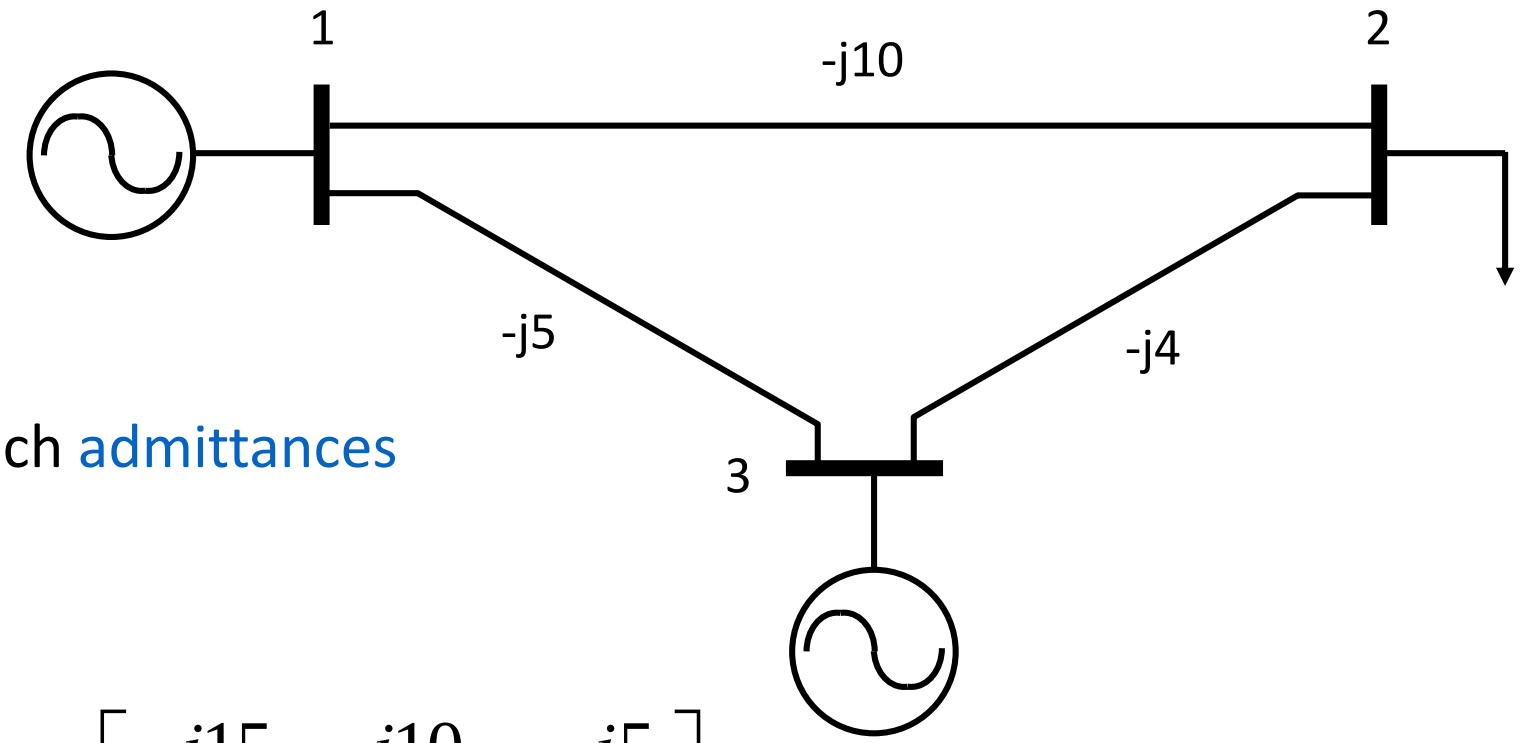


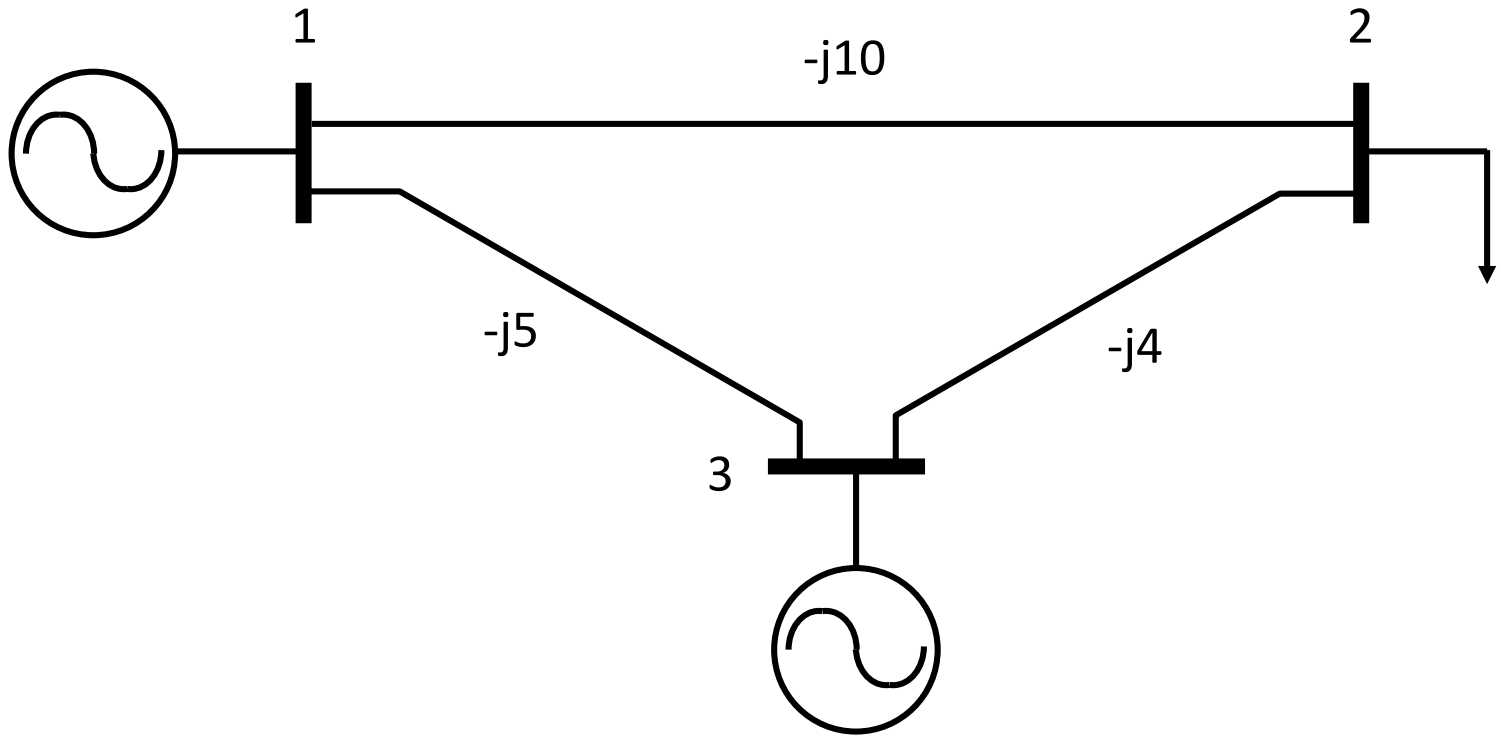
Figure shows branch admittances

$$Y = G + jB = \begin{bmatrix} -j15 & j10 & j5 \\ j10 & -j14 & j4 \\ j5 & j4 & -j9 \end{bmatrix}$$

Power Flow Equations

$$P_k = \sum_{i=1}^{N} V_k V_i [G_{ki} cos\theta_{ki} + B_{ki} sin\theta_{ki}]$$

$$Q_k = \sum_{i=1}^{N} V_k V_i [G_{ki} sin\theta_{ki} - B_{ki} cos\theta_{ki}]$$



- Two equations for each bus
  - First equation: balance of active power at that bus
  - Second equation: balance of reactive power at that bus
- Left hand side: power injected at the bus (> 0 for generation, < 0 for load)
- Right hand side: power flowing in the branches

Writing the equations for this system

$$P_k = \sum_{i=1}^{N} V_k V_i [G_{ki} cos\theta_{ki} + B_{ki} sin\theta_{ki}]$$

$$Q_k = \sum_{i=1}^{N} V_k V_i [G_{ki} sin\theta_{ki} - B_{ki} cos\theta_{ki}]$$

$$Y = G + jB = \begin{bmatrix} -j15 & j10 & j5 \\ j10 & -j14 & j4 \\ j5 & j4 & -j9 \end{bmatrix}$$

$$P_1 = -15V_1V_1 \sin q_{11} + 10V_1V_2 \sin q_{12} + 5V_1V_3 \sin q_{13}$$

$$Q_1 = +15V_1V_1 \cos q_{11} - 10V_1V_2 \cos q_{12} - 5V_1V_3 \cos q_{13}$$

$$P_2 = +10V_2V_1 \sin q_{21} - 14V_2V_2 \sin q_{22} + 4V_2V_3 \sin q_{23}$$

$$Q_2 = -10V_2V_1 \cos q_{21} + 14V_2V_2 \cos q_{22} - 4V_2V_3 \cos q_{23}$$

$$P_3 = +5V_3V_1 \sin q_{31} + 4V_3V_2 \sin q_{32} - 9V_3V_3 \sin q_{33}$$

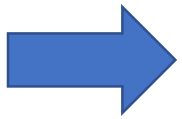
$$Q_3 = -5V_3V_1 \cos q_{31} - 4V_3V_2 \cos q_{32} + 9V_3V_3 \cos q_{33}$$

(All the G terms are zero because we neglected the resistance of the lines)

Simplifying the power flow equations

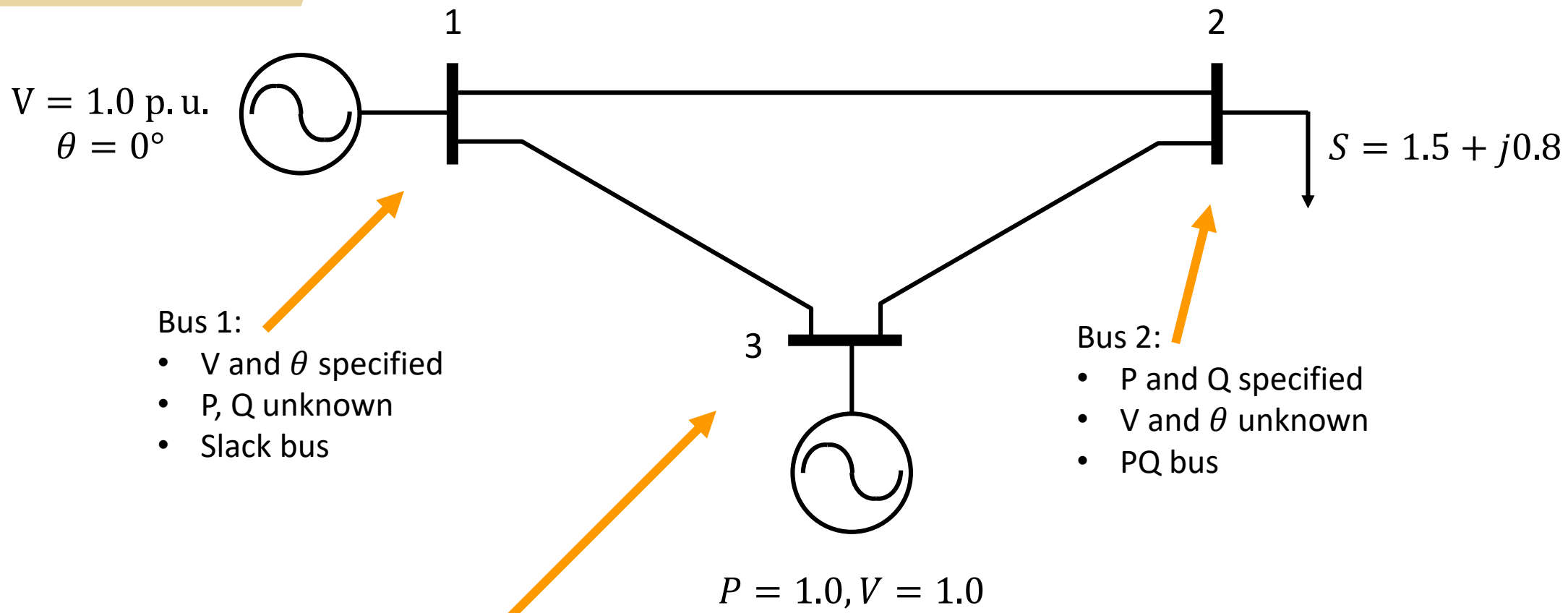
$$\sin\theta_{11} = \sin\theta_{22} = \sin\theta_{33} = \sin 0° = 0$$
$$\cos\theta_{11} = \cos\theta_{22} = \cos\theta_{33} = \cos 0° = 1$$

$$P_1 = -15V_1V_1\sin\theta_{11} + 10V_1V_2\sin\theta_{12} + 5V_1V_3\sin\theta_{13}$$
$$Q_1 = +15V_1V_1\cos\theta_{11} - 10V_1V_2\cos\theta_{12} - 5V_1V_3\cos\theta_{13}$$
$$P_2 = +10V_2V_1\sin\theta_{21} - 14V_2V_2\sin\theta_{22} + 4V_2V_3\sin\theta_{23}$$
$$Q_2 = -10V_2V_1\cos\theta_{21} + 14V_2V_2\cos\theta_{22} - 4V_2V_3\cos\theta_{23}$$
$$P_3 = +5V_3V_1\sin\theta_{31} + 4V_3V_2\sin\theta_{32} - 9V_3V_3\sin\theta_{33}$$
$$Q_3 = -5V_3V_1\cos\theta_{31} - 4V_3V_2\cos\theta_{32} + 9V_3V_3\cos\theta_{33}$$



$$P_1 = +10V_1V_2\sin\theta_{12} + 5V_1V_3\sin\theta_{13}$$
$$Q_1 = +15V_1^2 - 10V_1V_2\cos\theta_{12} - 5V_1V_3\cos\theta_{13}$$
$$P_2 = +10V_2V_1\sin\theta_{21} + 4V_2V_3\sin\theta_{23}$$
$$Q_2 = -10V_2V_1\cos\theta_{21} + 14V_2^2 - 4V_2V_3\cos\theta_{23}$$
$$P_3 = +5V_3V_1\sin\theta_{31} + 4V_3V_2\sin\theta_{32}$$
$$Q_3 = -5V_3V_1\cos\theta_{31} - 4V_3V_2\cos\theta_{32} + 9V_3^2$$

Bus types



Bus 1:
- V and θ specified
- P, Q unknown
- Slack bus

Bus 2:
- P and Q specified
- V and θ unknown
- PQ bus

Bus 3:
- P and V specified
- Q and θ unknown
- PV bus

Two groups of equations

Equations where we know the injections: implicit equations

$$P_2 = +10V_2V_1\sin\theta_{21} + 4V_2V_3\sin\theta_{23}$$ ⇒ P equation of PQ bus

$$Q_2 = -10V_2V_1\cos\theta_{21} + 14V_2^2 - 4V_2V_3\cos\theta_{23}$$ ⇒ Q equation of PQ bus

$$P_3 = +5V_3V_1\sin\theta_{31} + 4V_3V_2\sin\theta_{32}$$ ⇒ P equation of PV bus

Equations where we don't know the injections: explicit equations

$$P_1 = +10V_1V_2\sin\theta_{12} + 5V_1V_3\sin\theta_{13}$$ ⇒ P equation of slack bus

$$Q_1 = +15V_1^2 - 10V_1V_2\cos\theta_{12} - 5V_1V_3\cos\theta_{13}$$ ⇒ Q equation of slack bus

$$Q_3 = -5V_3V_1\cos\theta_{31} - 4V_3V_2\cos\theta_{32} + 9V_3^2$$ ⇒ Q equation of PV bus

Implicit Equations

Known variables: $P_2, Q_2, P_3, V_1, V_3, q_1$

Unknown variables: V_2, q_2, q_3

$$1.5 = +10V_2 \sin q_{21} + 4V_2 \sin q_{23}$$ ⇒ P equation of bus 2

$$0.8 = -10V_2 \cos q_{21} + 14V_2^2 - 4V_2 \cos q_{23}$$ ⇒ Q equation of bus 2

$$1.0 = +5 \sin q_{31} + 4V_2 \sin q_{32}$$ ⇒ P equation of bus 3

Set of simultaneous *non-linear* equations

We need a technique to solve such equations

Explicit Equations

Known variables: V_1, V_3, q_1

Solution of implicit equations: V_2, q_2, q_3

Unknown variables: P_1, Q_1, Q_3

$P_1 = +10V_2 \sin q_{12} + 5\sin q_{13}$ ⇒ P equation of bus 1

$Q_1 = +15 - 10V_2 \cos q_{12} - 5\cos q_{13}$ ⇒ Q equation of bus 1

$Q_3 = -5\cos q_{31} - 4V_2 \cos q_{32} + 9$ ⇒ Q equation of bus 3

Can calculate P_1, Q_1, Q_3 easily once we have solved the implicit equations to find V_2, q_2, q_3

Sparsity of the power flow equations

$$\begin{cases} P_k = \sum_{i=1}^{N} V_k V_i [G_{ki} cos\theta_{ki} + B_{ki} sin\theta_{ki}] \\ Q_k = \sum_{i=1}^{N} V_k V_i [G_{ki} sin\theta_{ki} - B_{ki} cos\theta_{ki}] \end{cases}$$

Equations for bus k have a term involving V_i only if $G_{ki} \neq 0$ or $B_{ki} \neq 0$

→Power flow equations have the same structure as the Y matrix

→Very sparse set of equations

Solving the power flow equations

- The power flow equations are
  - Implicit
  - Non-linear

→No direct, closed-form solution

→Need to use an iterative solution technique

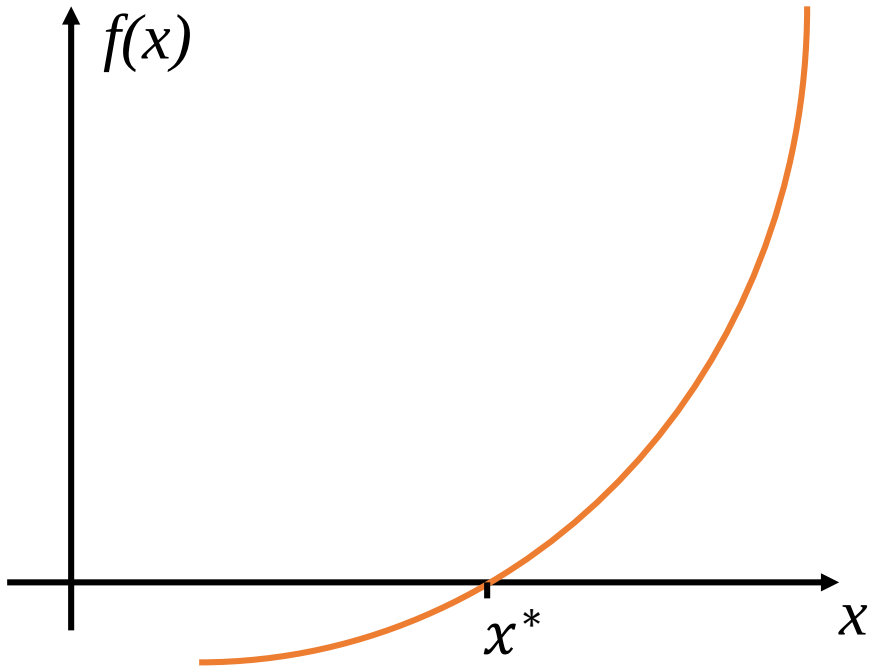
→Newton-Raphson method

  - Explain using single function of a single variable
  - Generalize to system of N functions of N variables

One-dimensional Newton-Raphson method

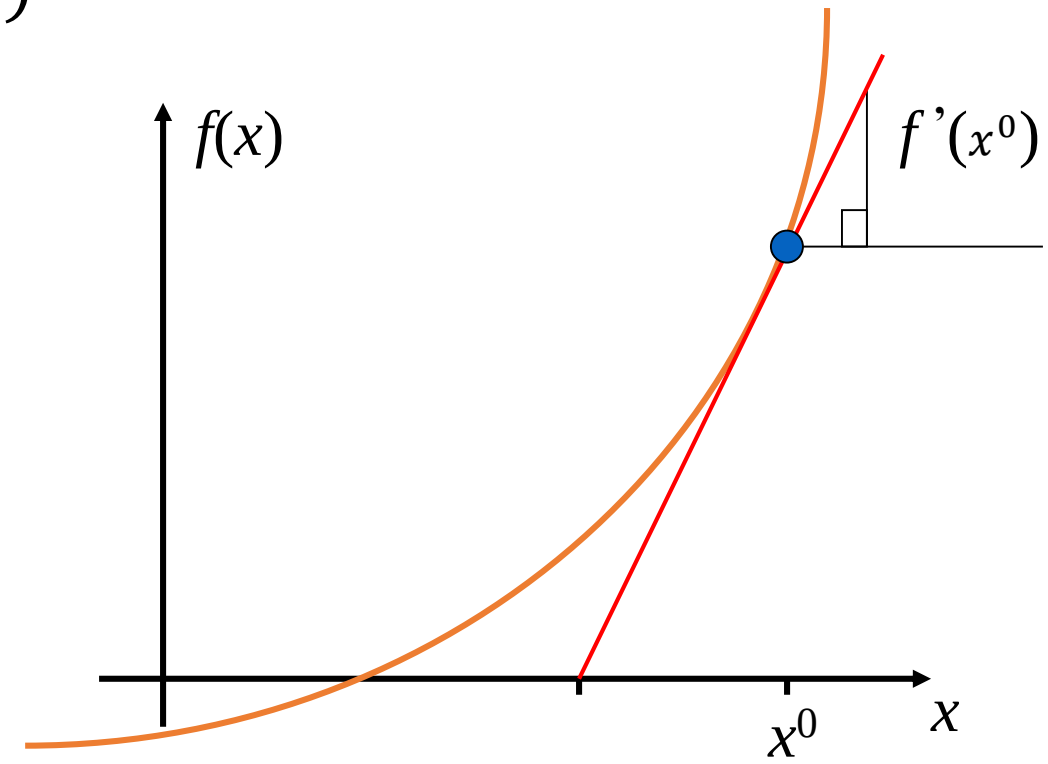
- Solve $f(x) = 0$ for x where $f(x)$ is a non-linear function of one variable

- Assume we know how to calculate:
  - $f(x)$ for any x
  - $f'(x) = \frac{df(x)}{dx}$ for any x



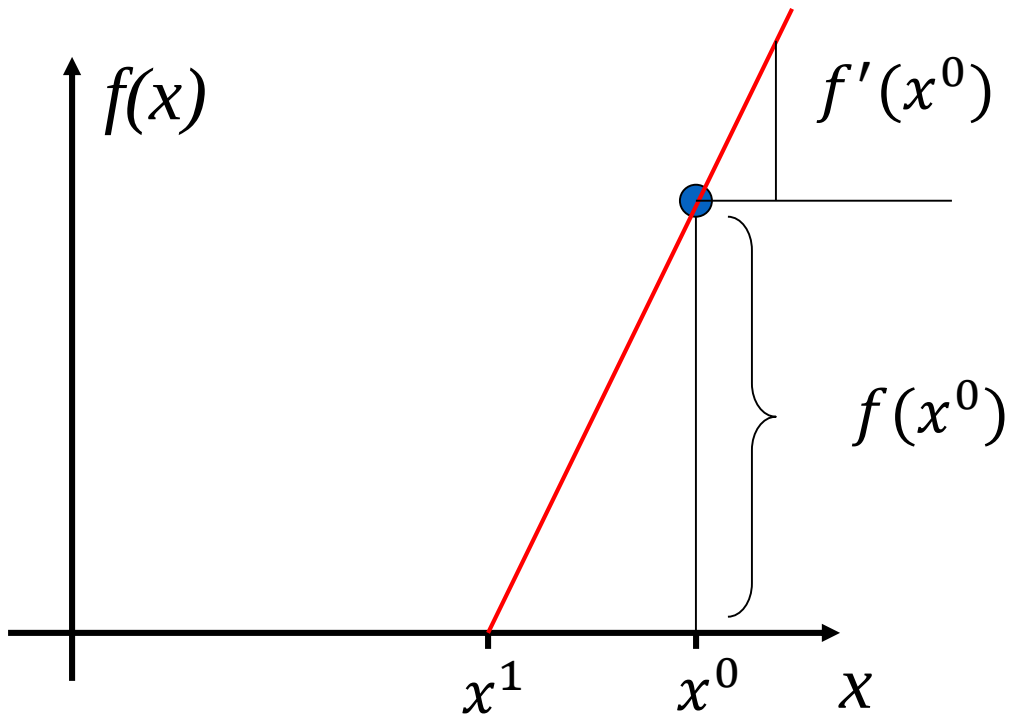
Linear approximation

- We can approximate a non-linear function $f(x)$ by its tangent at point x^0
- The slope of this tangent is equal to the derivative of the function, $f'(x)$ at point x^0: $f'(x^0)$



Using the linear approximation

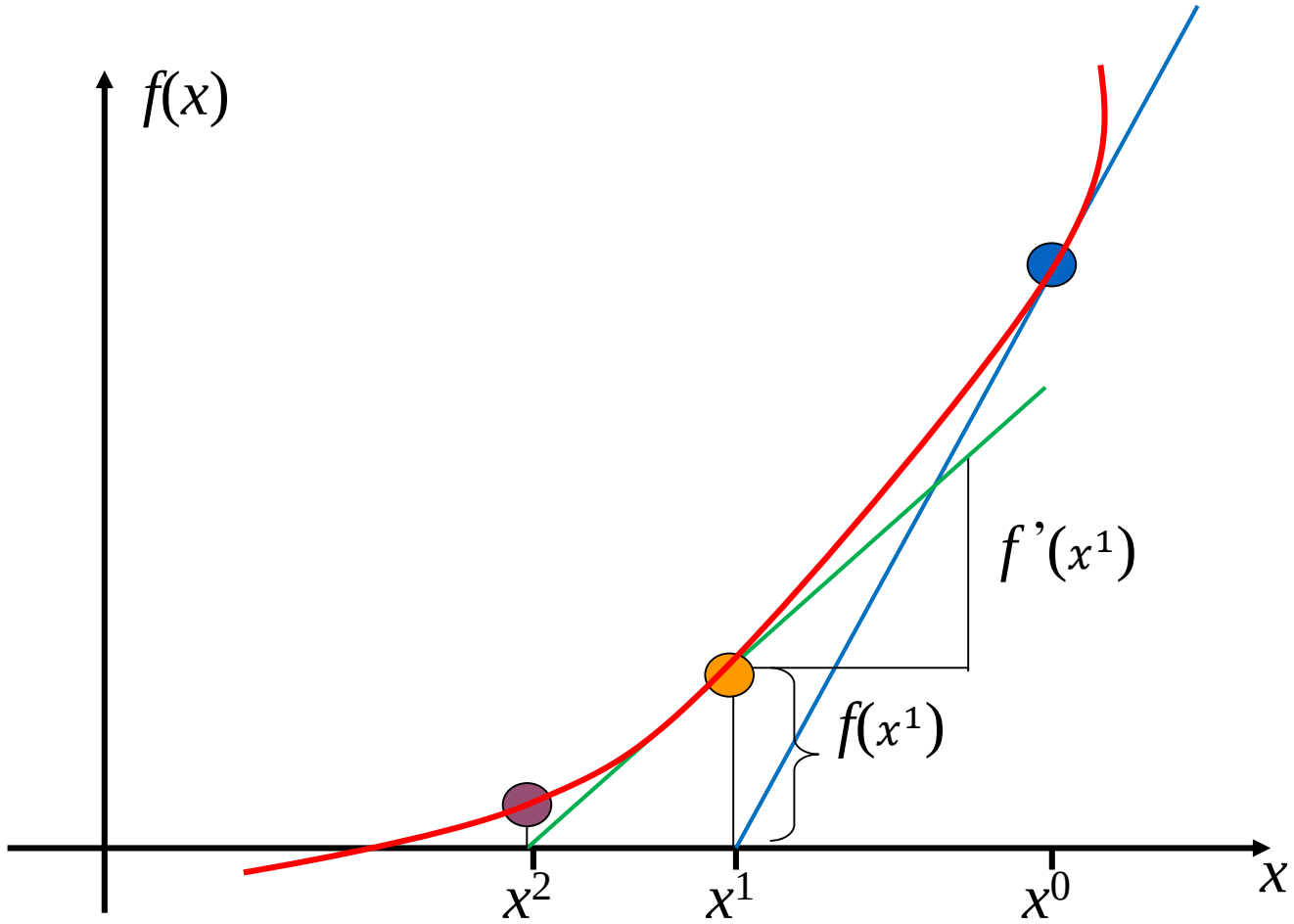
Assume that we can calculate $f(x)$ and $\frac{df(x)}{dx} = f'(x)$ for any x



$$f(x^0) = f'(x^0) \times (x^0\text{-}x^1)$$

$$x^1 = x^0 - \frac{f(x^0)}{f'(x^0)}$$

Repeated linearization



Formally: Taylor series expansion

Any continuous function can be expanded in a Taylor series:

$$f(x) = f(x^0) + f'(x^0)(x - x^0) + \frac{1}{2} f''(x^0)(x - x^0)^2 + \cdots$$

Linear approximation of this function:

$$f(x) \approx f(x^0) + f'(x^0)(x - x^0)$$

Replacing $f(x) = 0$ by its linear approximation:

$$f(x^0) + f'(x^0)(x - x^0) = 0$$

Solving this linear approximation:

$$x = x^0 - \frac{f(x^0)}{f'(x^0)}$$

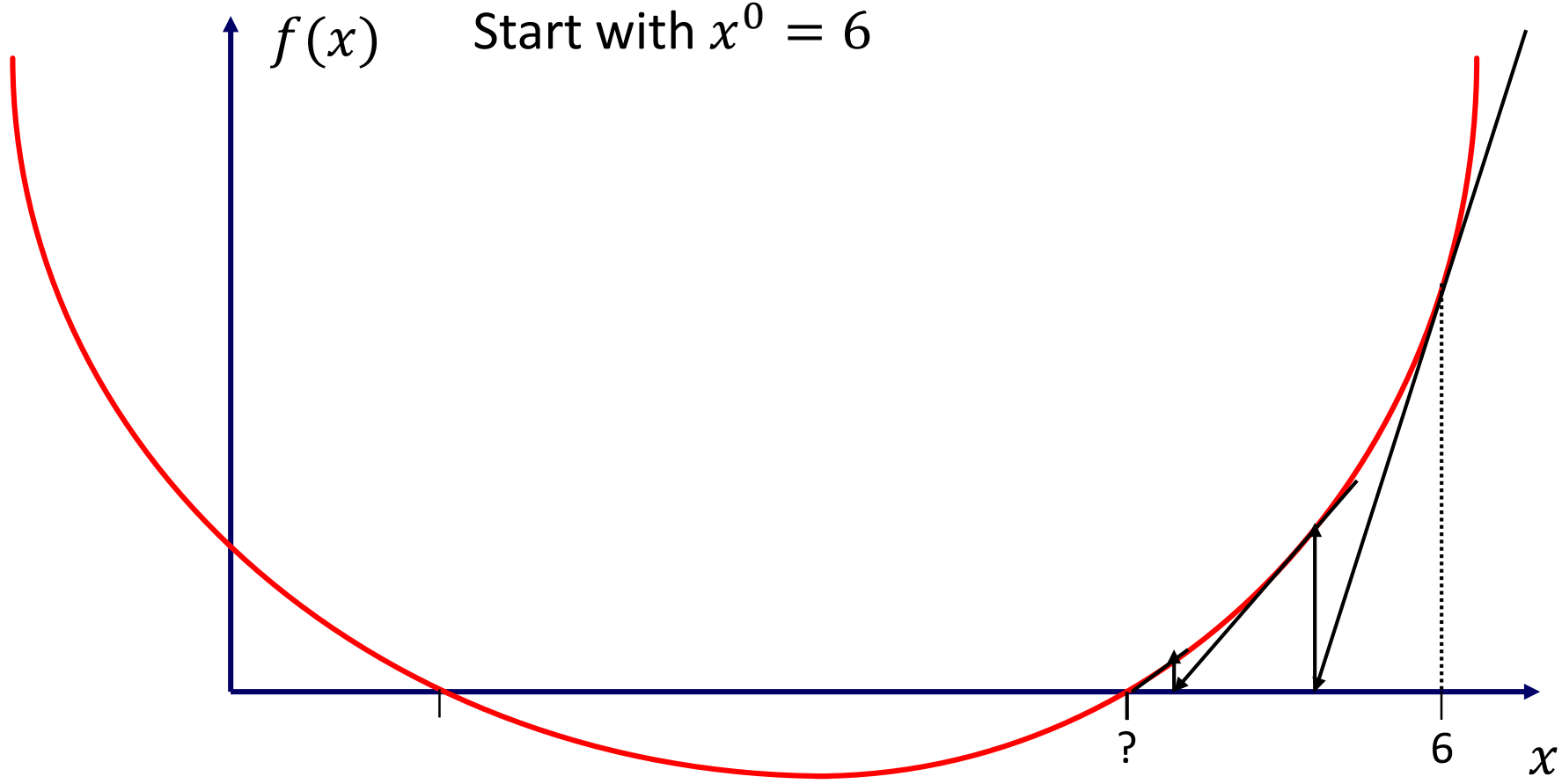
Solving by repeated linear approximation:

$$x^{r+1} = x^r - \frac{f(x^r)}{f'(x^r)} = x^r - \Delta x^r$$

Example

Find x such that $f(x) = x^2 - 5x + 4 = 0$

Start with $x^0 = 6$



Example

$f(x) = x^2 - 5x + 4 = 0$ $\qquad x^0 = 6$

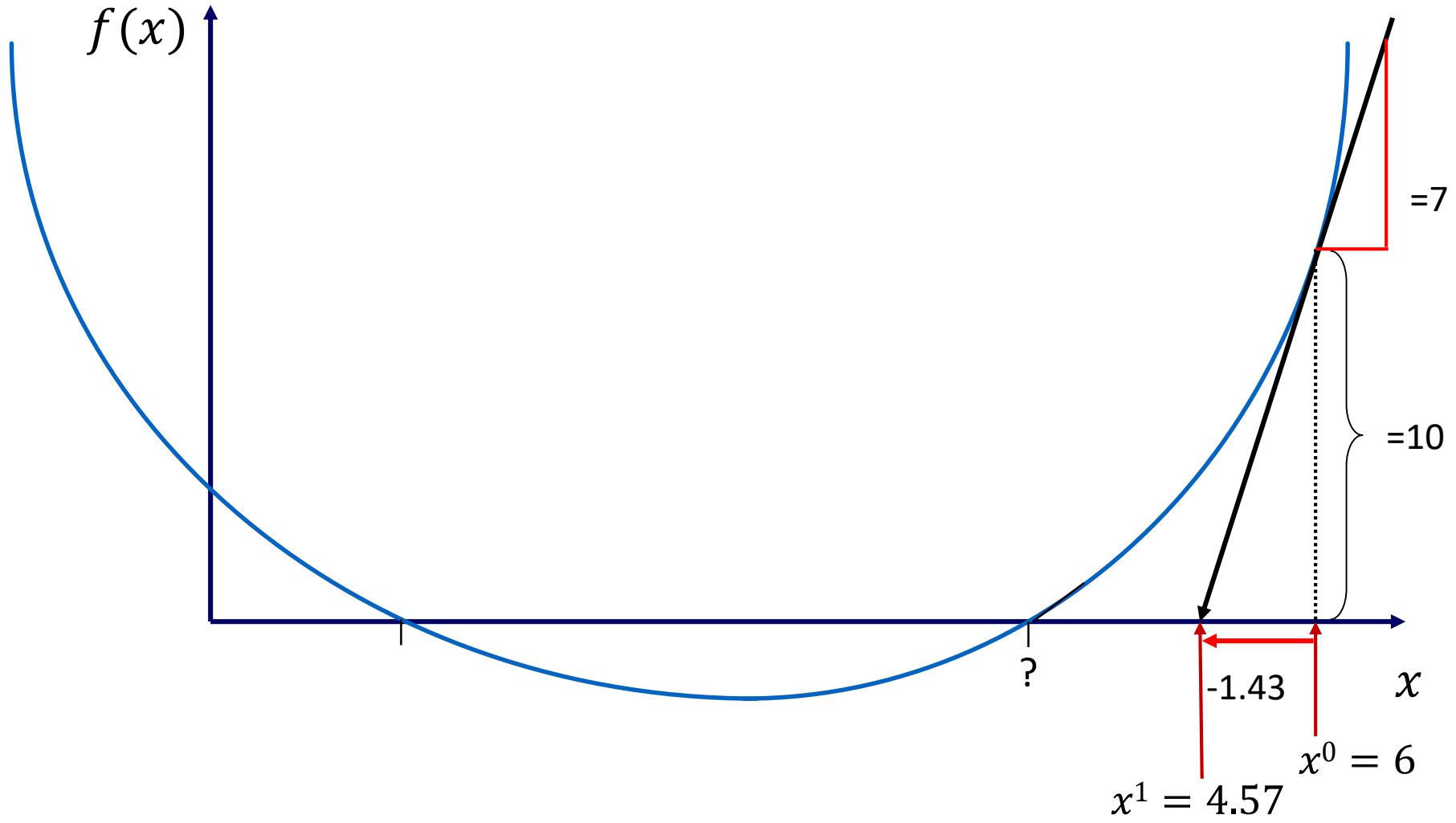
$$x^{k+1} = x^k - \frac{f(x^k)}{f'(x^k)} = x^k - \Delta x^k$$

$$f'(x) = 2x - 5$$

$$f(x^0) = 6^2 - 5 \times 6 + 4$$
$$= 10$$
$$f'(x^0) = 2 \times 6 - 5 = 7$$

$$\Delta x^0 = \frac{10}{7} = 1.43 \qquad x^1 = 6 - 1.43 = 4.57$$

Example



Example

$$f(x) = x^2 - 5x + 4 = 0$$

$$f'(x) = 2x - 5$$

$$x^1 = 4.57$$

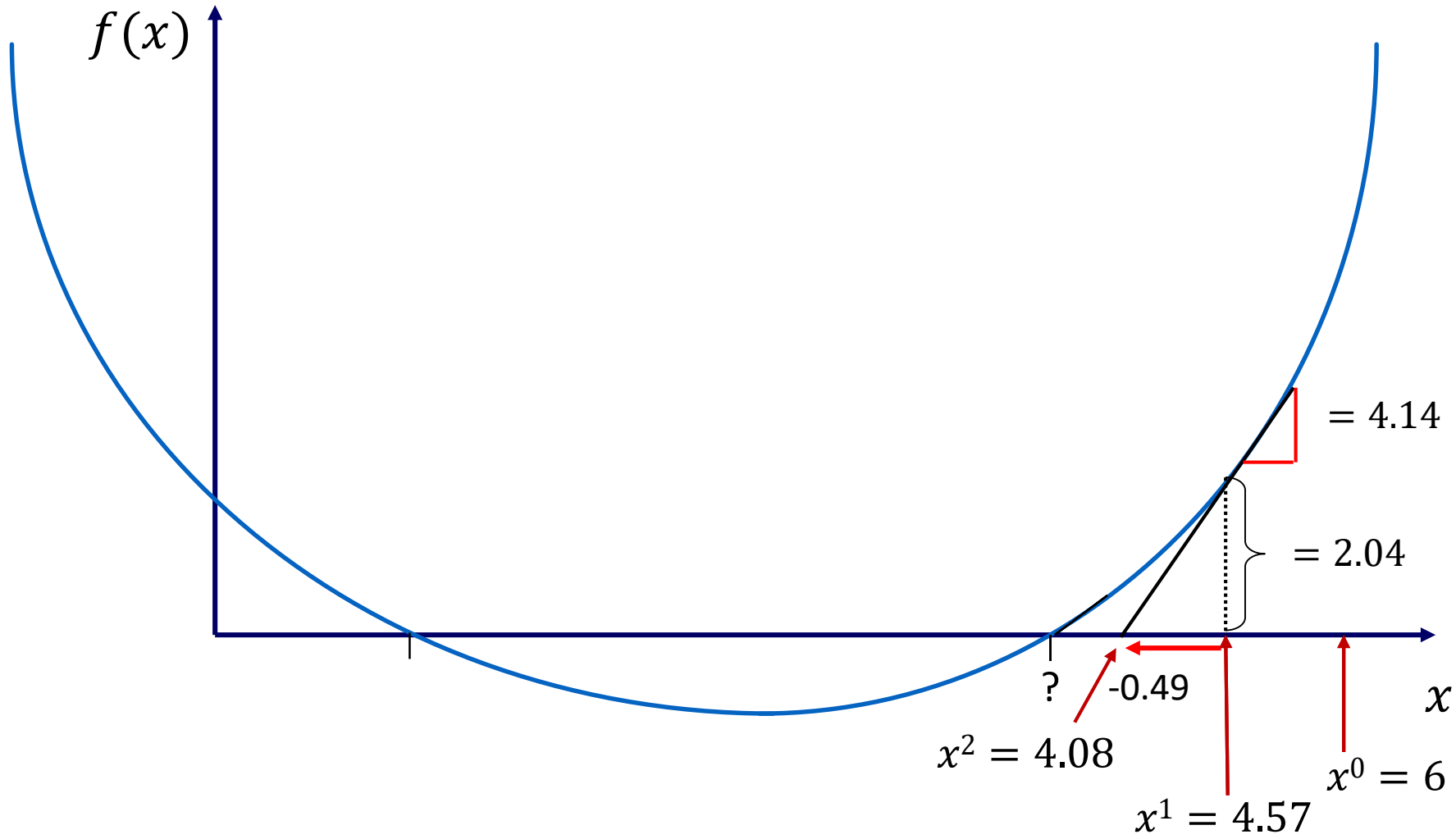
$$f(x^1) = 2.04$$

$$f'(x^1) = 4.14$$

$$\Delta x^1 = \frac{2.04}{4.14} = 0.49$$

$$x^2 = x^1 - \Delta x^1 = 4.57 - 0.49 = 4.08$$

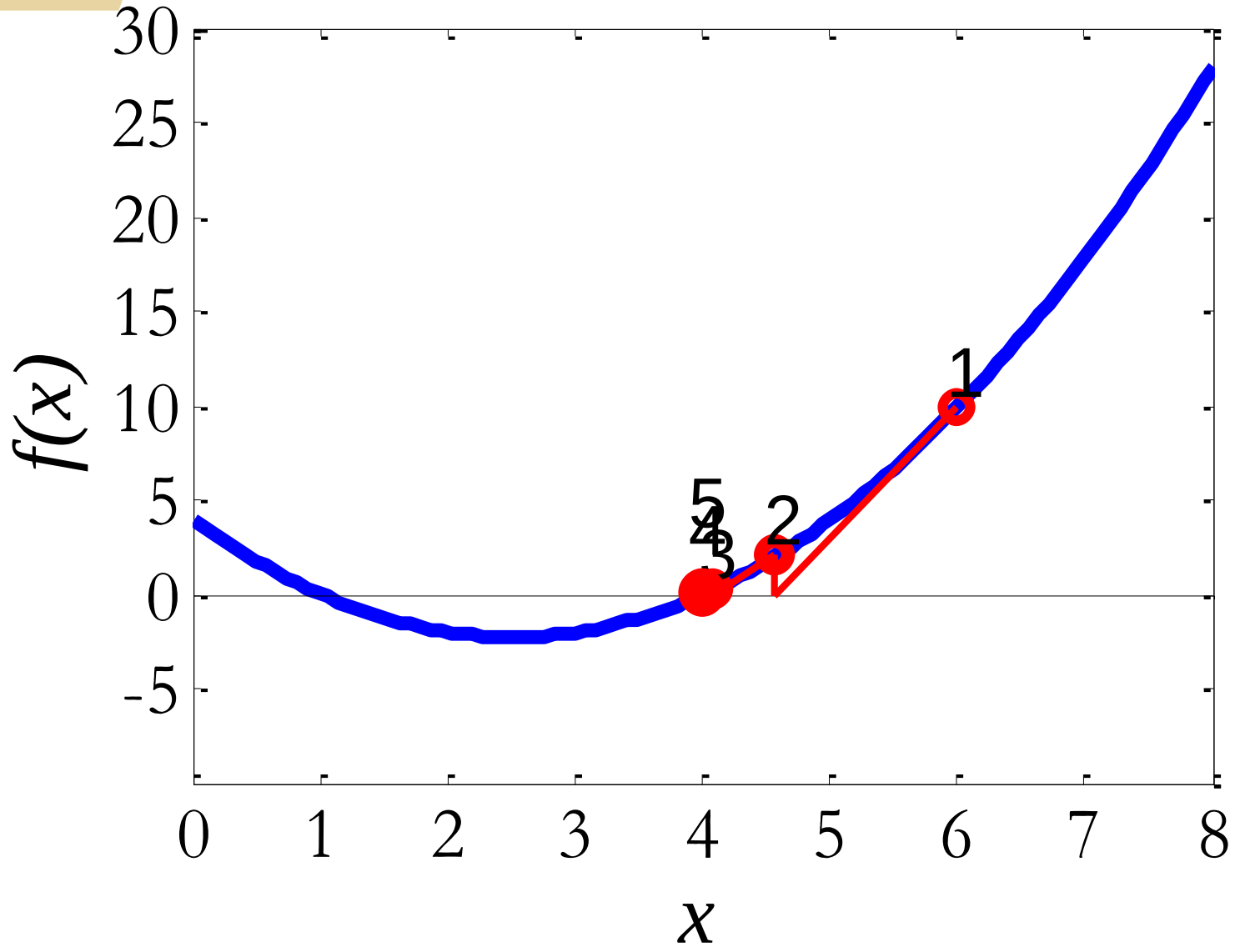
Example



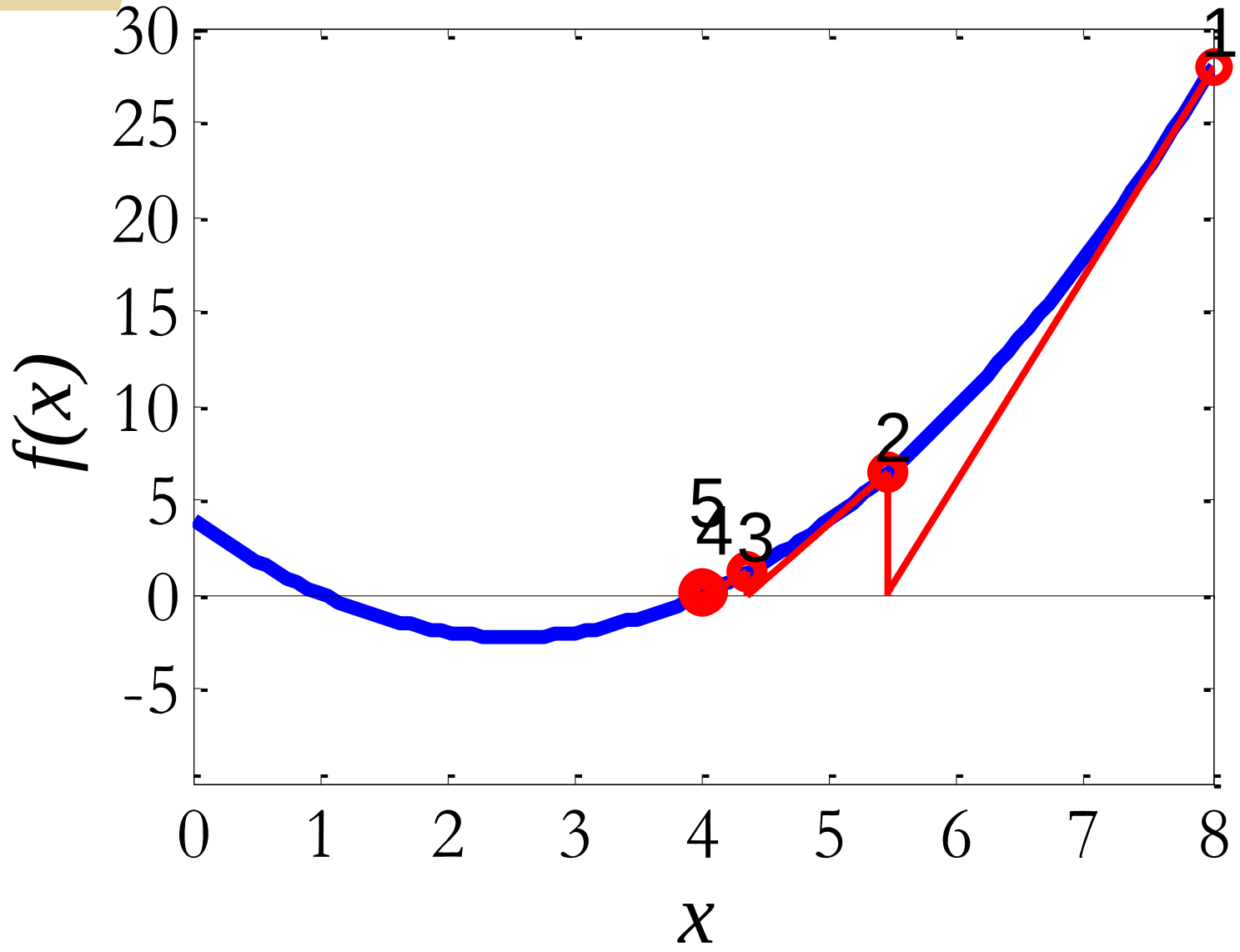
Example

r	x^{r}	$f(x^{r})$	$f'(x^{r})$	Δx^{r}	x^{r+1}
0	6.000	10.00	7.000	1.429	4.571
1	4.571	2.039	4.142	0.492	4.079
2	4.079	0.242	3.157	0.077	4.002
3	4.002	0.06	3.004	0.002	4.000

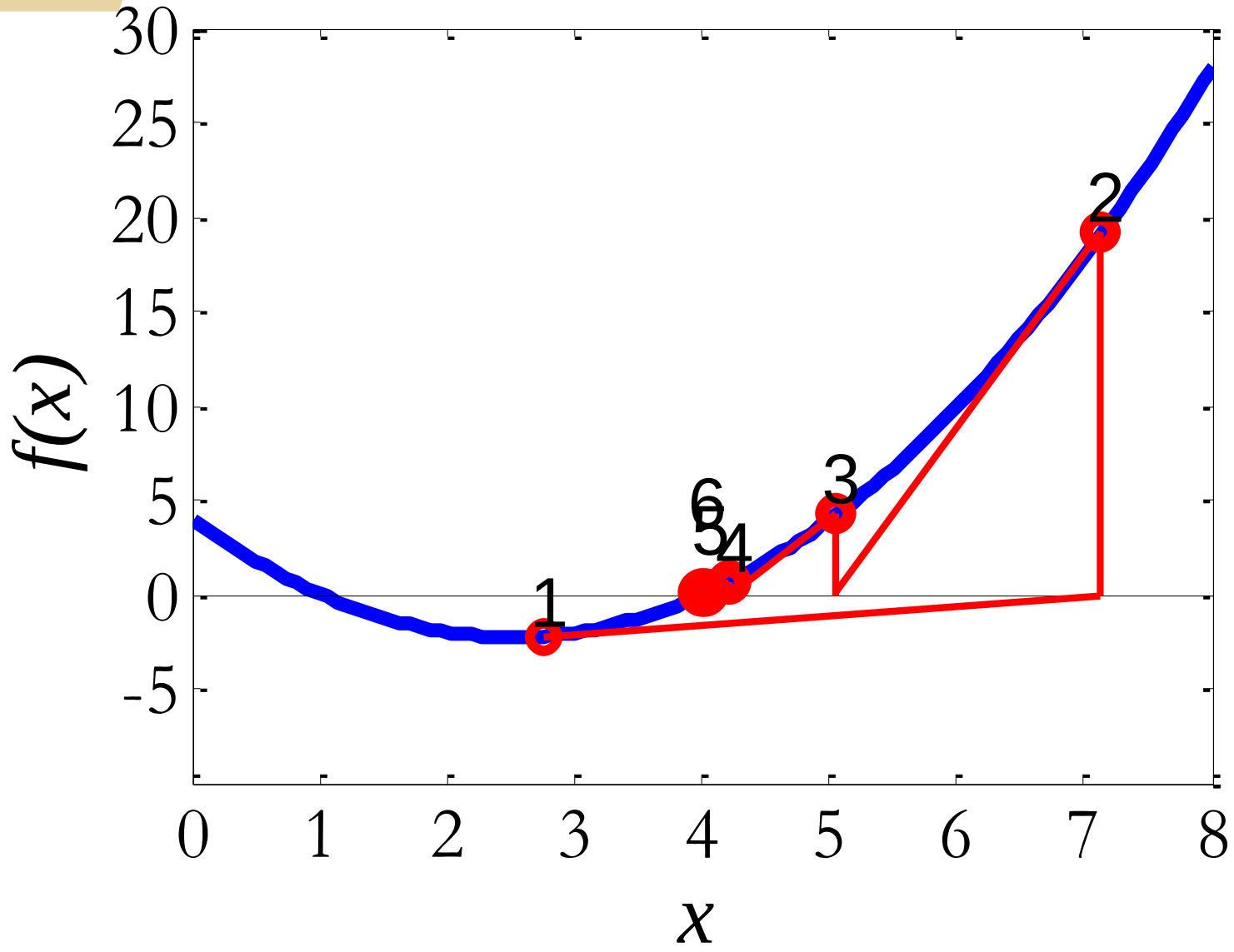
$x^0 = 6$



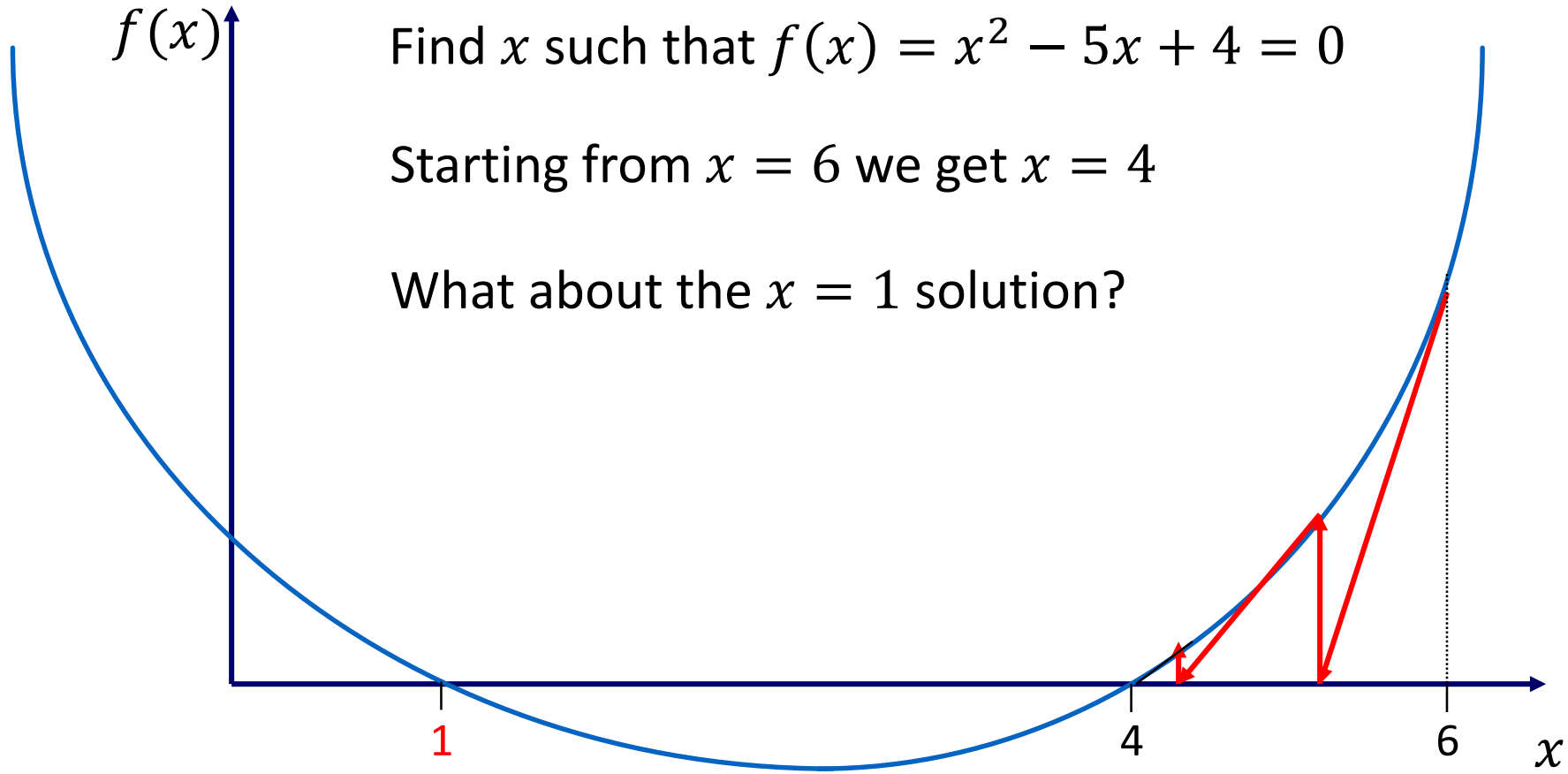
$x^0 = 8$



$x^0 = 2.75$



Example

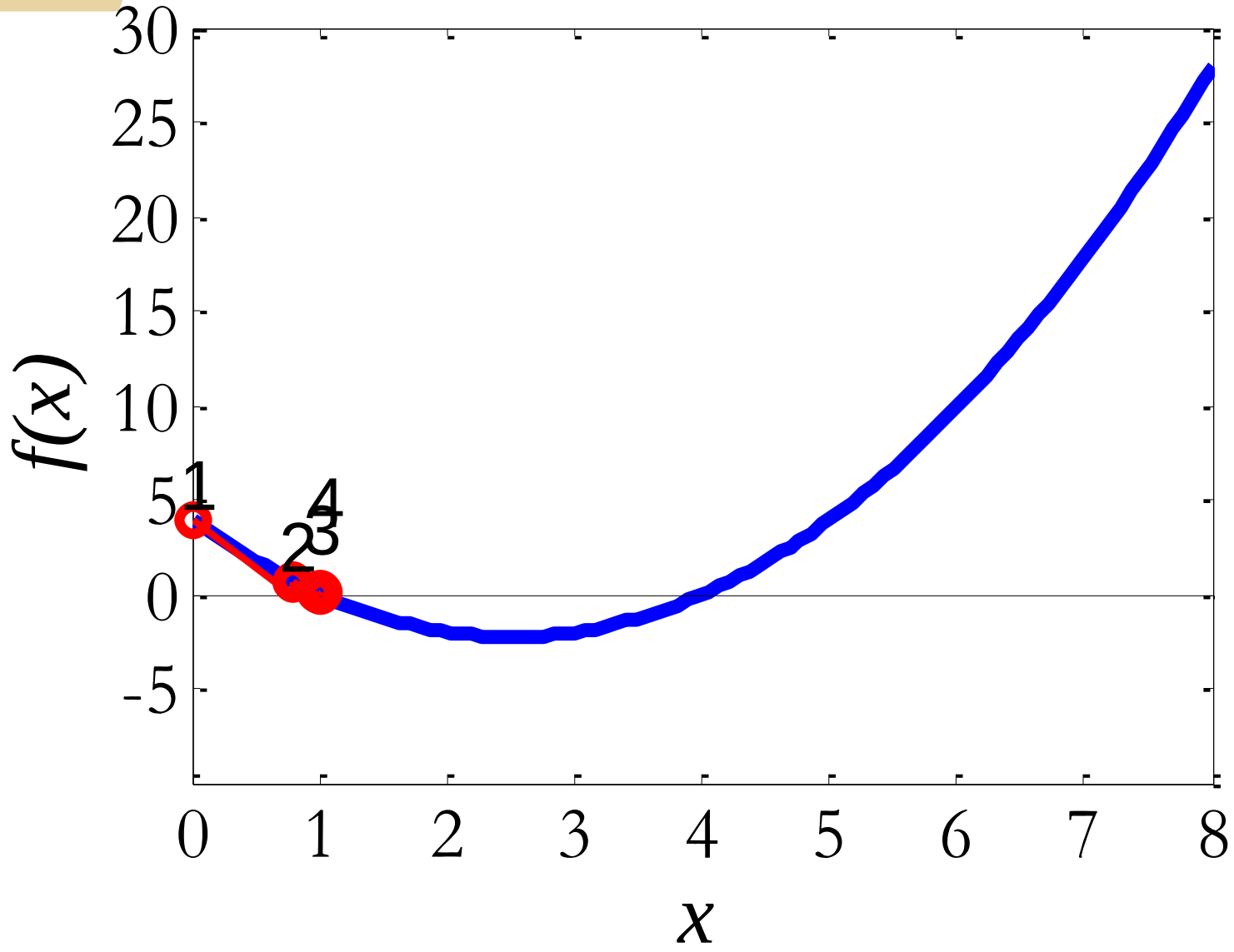


Find x such that $f(x) = x^2 - 5x + 4 = 0$

Starting from $x = 6$ we get $x = 4$

What about the $x = 1$ solution?

$x^0 = 0$



Convergence of iterative method

- Convergence is not guaranteed!
- Newton-Raphson method
  - Can converge to the “wrong” solution
  - Can diverge: iterative solution tends to infinity
  - Can exhibit oscillatory behavior
- An initial value close to the desired solution usually helps
- When the function is well-behaved, the Newton-Raphson method usually converges quickly

The power flow equations

$$P_k = \sum_{i=1}^{N} V_k V_i [G_{ki} cos\theta_{ki} + B_{ki} sin\theta_{ki}]$$

$$Q_k = \sum_{i=1}^{N} V_k V_i [G_{ki} sin\theta_{ki} - B_{ki} cos\theta_{ki}]$$

N buses → 2N equations

4N variables: P_k, Q_k, θ_k, V_k at each node k

Must have 2N unknown variables and 2N known variables

At each bus whether P_k, Q_k, θ_k, or V_k is known or unknown depends on the type of bus

Three bus types:

- PQ bus
- PV bus
- Slack bus

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One-dimensional Newton-Raphson (NR) method

$$x^{r+1} = x^r - \frac{f(x^r)}{f'(x^r)} = x^r - \Delta x^r$$

What about systems of N non-linear equations in N variables?

Application of NR method to two functions of two variables

Find x and y such that: $\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases}$

Taylor's series expansion:

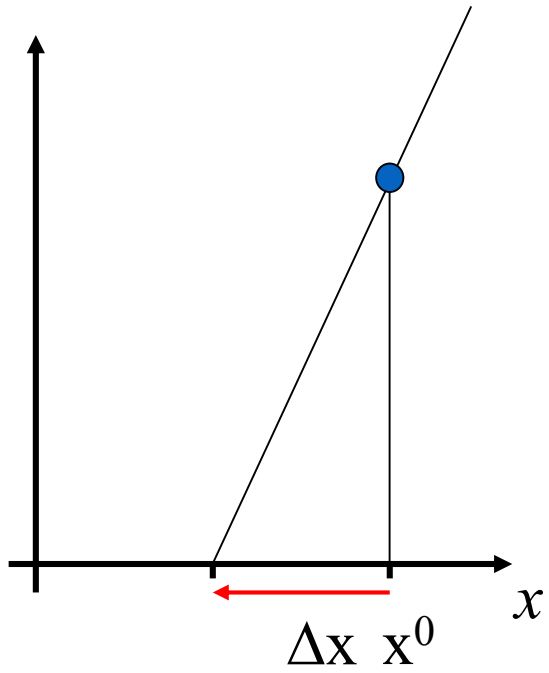
$$f(x,y) = f(x^0,y^0) + \frac{\partial f(x^0,y^0)}{\partial x}(x - x^0) + \frac{\partial f(x^0,y^0)}{\partial y}(y - y^0) + \cdots$$

$$g(x,y) = g(x^0,y^0) + \frac{\partial g(x^0,y^0)}{\partial x}(x - x^0) + \frac{\partial g(x^0,y^0)}{\partial y}(y - y^0) + \cdots$$

Let us neglect the higher order terms:

$$\begin{cases} f(x^0,y^0) + \frac{\partial f(x^0,y^0)}{\partial x}(x - x^0) + \frac{\partial f(x^0,y^0)}{\partial y}(y - y^0) = 0 \\ g(x^0,y^0) + \frac{\partial g(x^0,y^0)}{\partial x}(x - x^0) + \frac{\partial g(x^0,y^0)}{\partial y}(y - y^0) = 0 \end{cases}$$

Application of NR method to two functions of two variables



$$\begin{cases} f(x^0,y^0)+\dfrac{\partial f(x^0,y^0)}{\partial x}(x-x^0)+\dfrac{\partial f(x^0,y^0)}{\partial y}(y-y^0)=0 \\ g(x^0,y^0)+\dfrac{\partial g(x^0,y^0)}{\partial x}(x-x^0)+\dfrac{\partial g(x^0,y^0)}{\partial y}(y-y^0)=0 \end{cases}$$

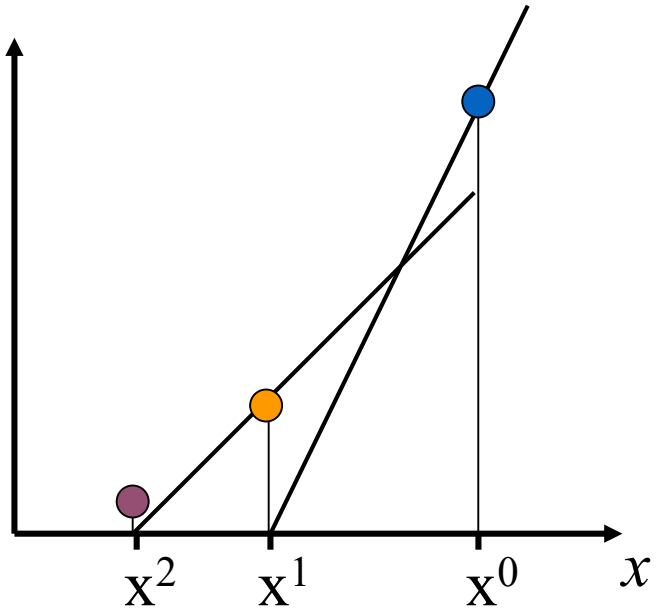
Rewrite in matrix form:

$$\begin{pmatrix} \dfrac{\partial f(x^0,y^0)}{\partial x} & \dfrac{\partial f(x^0,y^0)}{\partial y} \\ \dfrac{\partial g(x^0,y^0)}{\partial x} & \dfrac{\partial g(x^0,y^0)}{\partial y} \end{pmatrix} \begin{pmatrix} x-x^0 \\ y-y^0 \end{pmatrix} = -\begin{pmatrix} f(x^0,y^0) \\ g(x^0,y^0) \end{pmatrix}$$

Jacobian matrix $J(x,y)$

Application of NR method to two functions of two variables

$$J(x^0, y^0)\begin{pmatrix} x-x^0 \\ y-y^0 \end{pmatrix} = -\begin{pmatrix} f(x^0,y^0) \\ g(x^0,y^0) \end{pmatrix}$$



$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^0 \\ y^0 \end{pmatrix} - J(x^0, y^0)^{-1}\begin{pmatrix} f(x^0, y^0) \\ g(x^0, y^0) \end{pmatrix}$$

$$\begin{pmatrix} x^{k+1} \\ y^{k+1} \end{pmatrix} = \begin{pmatrix} x^k \\ y^k \end{pmatrix} - J(x^k, y^k)^{-1}\begin{pmatrix} f(x^k, y^k) \\ g(x^k, y^k) \end{pmatrix}$$

Example

Find x and y such that: $\begin{cases} f(x,y) \equiv x + y - 15 = 0 \\ g(x,y) \equiv x \cdot y - 50 = 0 \end{cases}$

Start from: $x^0 = 4 \; ; \; y^0 = 9$

Iterative expression:

$$\begin{pmatrix} x^{k+1} \\ y^{k+1} \end{pmatrix} = \begin{pmatrix} x^k \\ y^k \end{pmatrix} - J(x^k, y^k)^{-1} \begin{pmatrix} f(x^k, y^k) \\ g(x^k, y^k) \end{pmatrix}$$

$$\boldsymbol{J} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ y & x \end{pmatrix}$$

Inverting a 2x2 matrix

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad J^{-1} = \frac{1}{ad - cb} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

First iteration

$$\begin{cases} f(x,y) \equiv x + y - 15 = 0 \\ g(x,y) \equiv x \cdot y - 50 = 0 \end{cases}$$

$$J = \begin{pmatrix} 1 & 1 \\ y & x \end{pmatrix}$$

$$x^0 = 4 \; ; \; y^0 = 9$$

$$\begin{cases} f(x^0,y^0) \equiv x^0 + y^0 - 15 = -2 \\ g(x^0,y^0) \equiv x^0 \cdot y^0 - 50 = -14 \end{cases}$$

$$J(x^0,y^0) = \begin{pmatrix} 1 & 1 \\ 9 & 4 \end{pmatrix}$$

$$J^{-1}(x^0,y^0) = \begin{pmatrix} -0.8 & 0.2 \\ 1.8 & -0.2 \end{pmatrix}$$

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = - J^{-1}(x^0,y^0) \begin{pmatrix} f(x^0,y^0) \\ g(x^0,y^0) \end{pmatrix}$$

$$= - \begin{pmatrix} -0.8 & 0.2 \\ 1.8 & -0.2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -14 \end{pmatrix} = \begin{pmatrix} 1.2 \\ 0.8 \end{pmatrix}$$

$$\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} = \begin{pmatrix} x^0 \\ y^0 \end{pmatrix} + \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 9 \end{pmatrix} + \begin{pmatrix} 1.2 \\ 0.8 \end{pmatrix}$$

$$= \begin{pmatrix} 5.2 \\ 9.8 \end{pmatrix}$$

Second iteration

$$\begin{cases} f(x,y) \equiv x + y - 15 = 0 \\ g(x,y) \equiv x \cdot y - 50 = 0 \end{cases}$$

$$J = \begin{pmatrix} 1 & 1 \\ y & x \end{pmatrix}$$

$$x^1 = 5.2 \; ; \; y^1 = 9.8$$

$$\begin{cases} f(x^1,y^1) \equiv x^1 + y^1 - 15 = 0 \\ g(x^1,y^1) \equiv x^1 \cdot y^1 - 50 = 0.96 \end{cases}$$

$$J(x^1,y^1) = \begin{pmatrix} 1 & 1 \\ 9.8 & 5.2 \end{pmatrix}$$

$$J^{-1}(x^1,y^1) = \begin{pmatrix} -1.13 & 0.217 \\ 2.13 & -0.217 \end{pmatrix}$$

$$\begin{pmatrix} Dx \\ Dy \end{pmatrix} = - J^{-1}(x^1,y^1) \begin{pmatrix} f(x^1,y^1) \\ g(x^1,y^1) \end{pmatrix}$$

$$= - \begin{pmatrix} -1.13 & 0.217 \\ 2.13 & -0.217 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0.96 \end{pmatrix} = \begin{pmatrix} -0.209 \\ 0.209 \end{pmatrix}$$

$$\begin{pmatrix} x^2 \\ y^2 \end{pmatrix} = \begin{pmatrix} 5.2 \\ 9.8 \end{pmatrix} + \begin{pmatrix} -0.209 \\ 0.209 \end{pmatrix}$$

$$= \begin{pmatrix} 4.991 \\ 10.009 \end{pmatrix}$$

Third iteration

$$\begin{cases} f(x,y) \equiv x + y - 15 = 0 \\ g(x,y) \equiv x \cdot y - 50 = 0 \end{cases}$$

$$J = \begin{pmatrix} 1 & 1 \\ y & x \end{pmatrix}$$

$$x^2 = 4.991 \; ; \; y^2 = 10.009$$

$$\begin{cases} f(x^2,y^2) \equiv x^2 + y^2 - 15 = 0 \\ g(x^2,y^2) \equiv x^2 \cdot y^2 - 50 = -0.045 \end{cases}$$

$$J(x^2,y^2) = \begin{pmatrix} 1 & 1 \\ 10.009 & 4.991 \end{pmatrix}$$

$$J^{-1}(x^2,y^2) = \begin{pmatrix} -0.995 & 0.1993 \\ 1.995 & -0.199 \end{pmatrix}$$

$$\begin{pmatrix} x^3 \\ y^3 \end{pmatrix} = \begin{pmatrix} 4.991 \\ 10.009 \end{pmatrix} + \begin{pmatrix} 0.009 \\ -0.009 \end{pmatrix}$$

$$\begin{pmatrix} Dx \\ Dy \end{pmatrix} = - J^{-1}(x^2,y^2) \begin{pmatrix} f(x^2,y^2) \\ g(x^2,y^2) \end{pmatrix} \quad \Rightarrow \quad = \begin{pmatrix} 5.0 \\ 10.0 \end{pmatrix}$$

$$= - \begin{pmatrix} -0.995 & 0.1993 \\ 1.995 & -0.199 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -0.045 \end{pmatrix} = \begin{pmatrix} 0.009 \\ -0.009 \end{pmatrix}$$

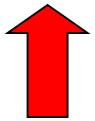
Fourth iteration

$$\begin{cases} f(x,y) \equiv x + y - 15 = 0 \\ g(x,y) \equiv x \cdot y - 50 = 0 \end{cases}$$

$$J = \begin{pmatrix} 1 & 1 \\ y & x \end{pmatrix}$$

$$x^3 = 5.0 \; ; \; y^3 = 10.0$$

$$\begin{cases} f(x^3,y^3) \equiv x^3 + y^3 - 15 = 0 \\ g(x^3,y^3) \equiv x^3 \cdot y^3 - 50 = 0 \end{cases}$$



Stop because we have found a solution

- In general, we would not find an exact solution.
- We would stop when the value of the functions is sufficiently small.

Newton Raphson method

Given a set of non-linear equations:

$$\begin{cases} f(x,y)=0 \\ g(x,y)=0 \end{cases}$$

1. Linearize around (x^k, y^k)

$$J_{(x^k,y^k)} \cdot \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = - \begin{pmatrix} f(x^k,y^k) \\ g(x^k,y^k) \end{pmatrix}$$

Where the Jacobian matrix is:

$$J_{(x^k,y^k)} = \begin{pmatrix} \frac{\partial f(x^k,y^k)}{\partial x} & \frac{\partial f(x^k,y^k)}{\partial y} \\ \frac{\partial g(x^k,y^k)}{\partial x} & \frac{\partial g(x^k,y^k)}{\partial y} \end{pmatrix}$$

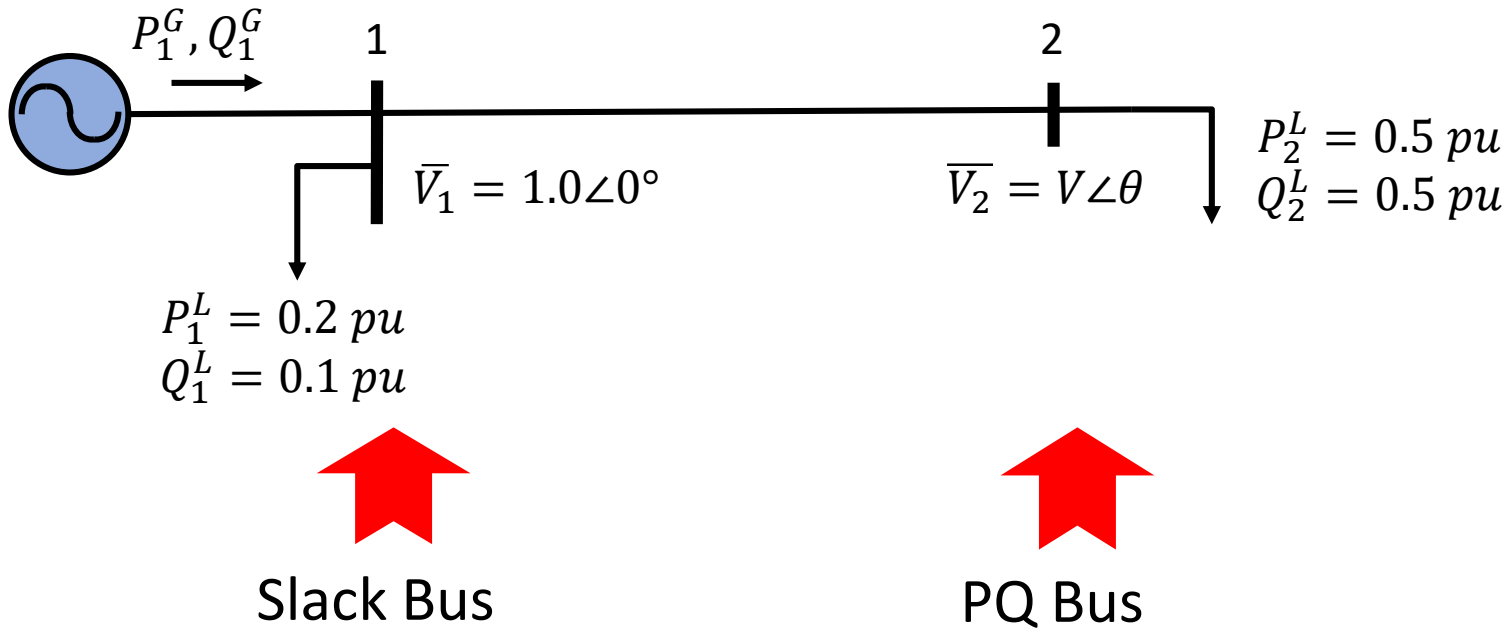
2. Calculate correction

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = - J^{-1}_{(x^k,y^k)} \begin{pmatrix} f(x^k,y^k) \\ g(x^k,y^k) \end{pmatrix}$$

3. Repeat until convergence

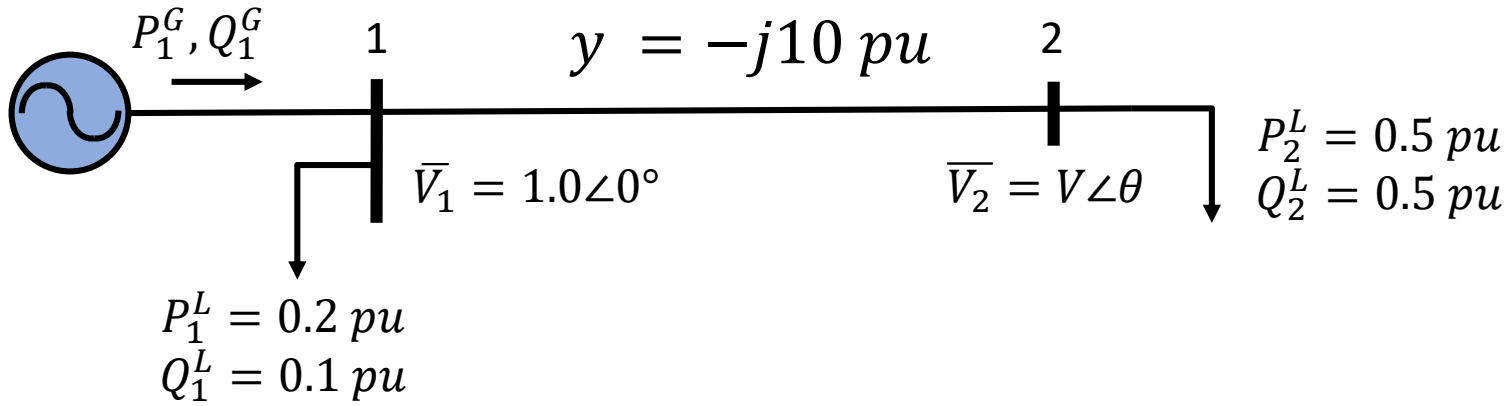
$$\begin{pmatrix} x^{k+1} \\ y^{k+1} \end{pmatrix} = \begin{pmatrix} x^k \\ y^k \end{pmatrix} + \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

Example: Two-bus power flow



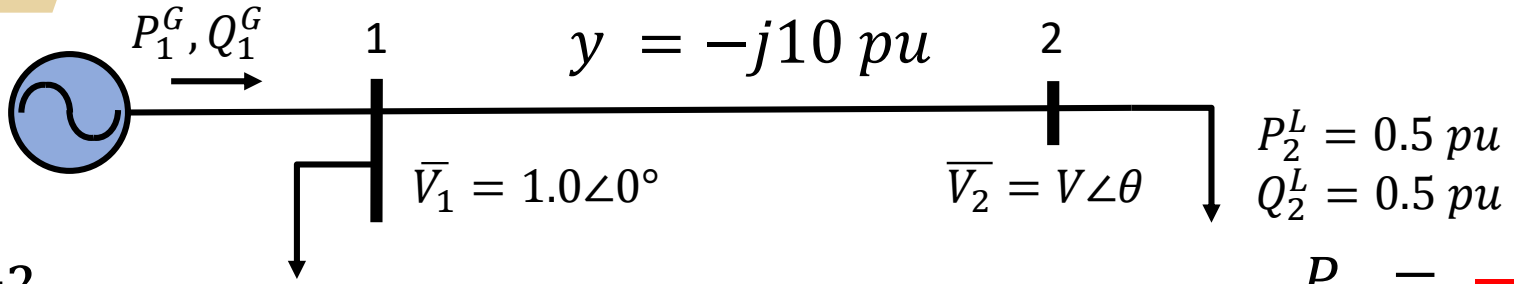
Note: we can't solve this problem directly because the powers are given at one end and the voltage magnitude at the other.

Example: Two-bus power flow



$$Y = G + jB = j\begin{pmatrix} -10 & 10 \\ 10 & -10 \end{pmatrix}$$

Example: Two-bus power flow



$$P_2 = -0.5\ pu$$
$$Q_2 = -0.5\ pu$$

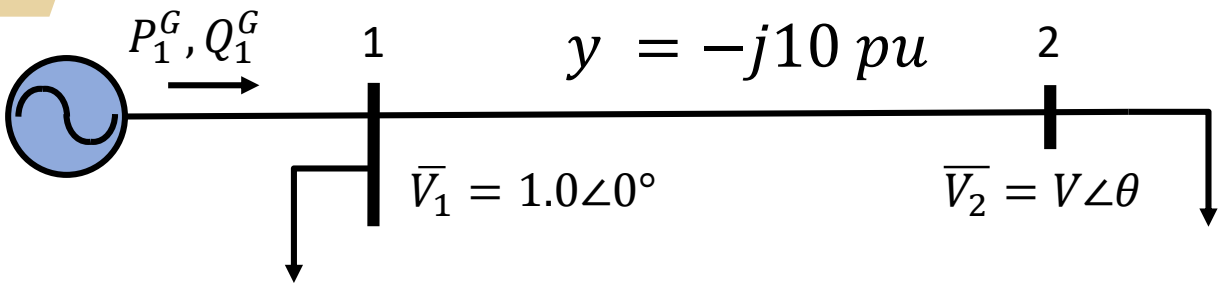
$$P_k = \sum_{i=1}^{2} V_k V_i [G_{ki} cos\theta_{ki} + B_{ki} sin\theta_{ki}]$$

$$Q_k = \sum_{i=1}^{2} V_k V_i [G_{ki} sin\theta_{ki} - B_{ki} cos\theta_{ki}]$$

$$\begin{cases} P_k = \sum_{i=1}^{2} V_k V_i B_{ki} sin\theta_{ki} \\ Q_k = \sum_{i=1}^{2} -V_k V_i B_{ki} cos\theta_{ki} \end{cases}$$

$$Y = G + jB = j\begin{pmatrix} -10 & 10 \\ 10 & -10 \end{pmatrix} \rightarrow G_{ki} = 0\ \forall\ k, i$$

Example: Two-bus power flow



$P_1 = ?$ $\quad$ $P_2 = -0.5\ pu$

$Q_1 = ?$ $\quad$ $Q_2 = -0.5\ pu$

$$P_k = \sum_{i=1}^{2} V_k V_i B_{ki} sin\theta_{ki}$$

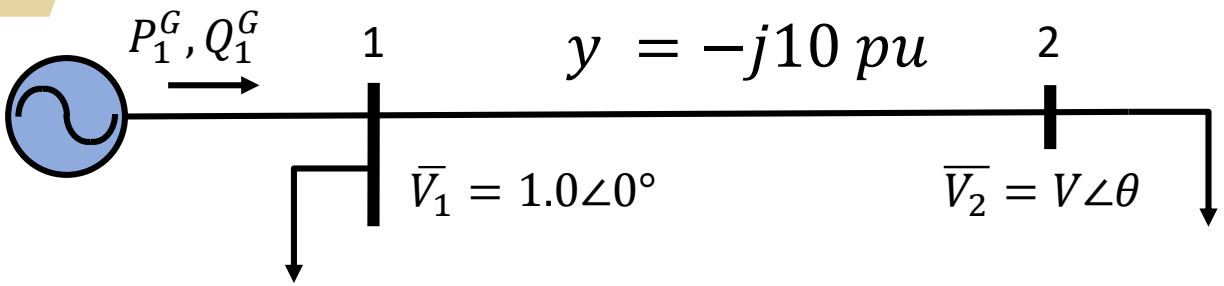
$$k = 1$$

$$P_1 = V_1 V_1 B_{11} sin\theta_{11} + V_1 V_2 B_{12} sin\theta_{12}$$

$$Q_k = \sum_{i=1}^{2} -V_k V_i B_{ki} cos\theta_{ki}$$

$$Q_1 = -V_1 V_1 B_{11} cos\theta_{11} - V_1 V_2 B_{12} cos\theta_{12}$$

Example: Two-bus power flow



$P_1 = ?$ $\quad P_2 = -0.5\ pu$

$Q_1 = ?$ $\quad Q_2 = -0.5\ pu$

$$P_k = \sum_{i=1}^{2} V_k V_i B_{ki} sin\theta_{ki}$$

$$P_2 = V_2 V_1 B_{21} sin\theta_{21} + V_2 V_2 B_{22} sin\theta_{22}$$

$$k = 2$$

$$Q_k = \sum_{i=1}^{2} -V_k V_i B_{ki} cos\theta_{ki}$$

$$Q_2 = -V_2 V_1 B_{21} cos\theta_{21} - V_2 V_2 B_{22} cos\theta_{22}$$

Example: Two-bus power flow

$$\begin{cases} P_1 = V_1 V_1 B_{11} sin\theta_{11} + V_1 V_2 B_{12} sin\theta_{12} \\ Q_1 = -V_1 V_1 B_{11} cos\theta_{11} - V_1 V_2 B_{12} cos\theta_{12} \\ P_2 = V_2 V_1 B_{21} sin\theta_{21} + V_2 V_2 B_{22} sin\theta_{22} \\ Q_2 = -V_2 V_1 B_{21} cos\theta_{21} - V_2 V_2 B_{22} cos\theta_{22} \end{cases}$$

$$\begin{cases} P_1 = -V B_{12} sin\theta \\ Q_1 = -B_{11} - V B_{12}\, cos\theta \\ P_2 = V B_{21} sin\theta \\ Q_2 = -V B_{21} cos\theta - V^2 B_{22} \end{cases}$$

$$\theta_{11} = \theta_{22} = 0 \rightarrow sin\theta_{11} = sin\theta_{22} = 0$$

$$\theta_{11} = \theta_{22} = 0 \rightarrow cos\theta_{11} = cos\theta_{22} = 1$$

$$\theta_1 = 0° \rightarrow \theta_{21} = \theta; \theta_{12} = -\theta \rightarrow sin\,\theta_{12} = -sin\theta$$

$$V_1 = 1; V_2 = V$$

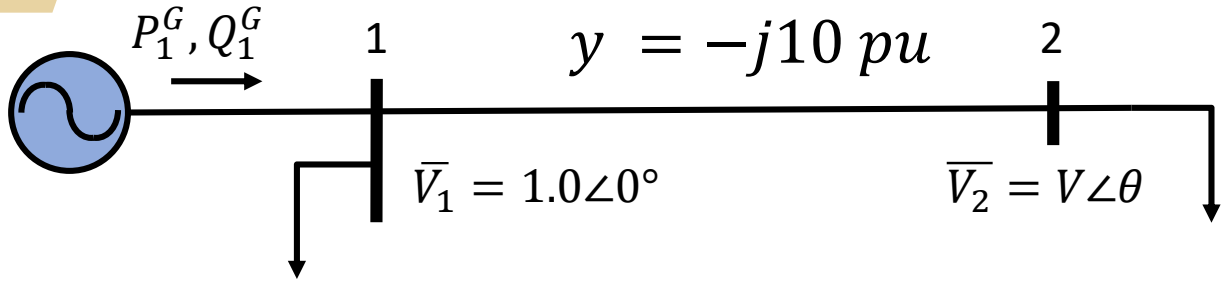
Example: Two-bus power flow

$$\begin{cases} P_1 = -VB_{12}sin\theta \\ Q_1 = -B_{11} - VB_{12}\,cos\theta \\ P_2 = VB_{21}sin\theta \\ Q_2 = -VB_{21}cos\theta - V^2B_{22} \end{cases}$$

$$\begin{cases} P_1 = -10\,V\,sin\theta \\ Q_1 = 10 - 10\,V\,cos\theta \\ P_2 = 10\,V\,sin\theta \\ Q_2 = -10\,V\,cos\theta + 10\,V^2 \end{cases}$$

$$B = \begin{pmatrix} -10 & 10 \\ 10 & -10 \end{pmatrix}$$

Example: Two-bus power flow



$P_1 = ?$
$Q_1 = ?$

$P_2 = -0.5\ pu$
$Q_2 = -0.5\ pu$

$$P_1 = -10\ V\ sin\theta$$
$$Q_1 = 10 - 10\ V\ cos\theta$$
$$P_2 = 10\ V\ sin\theta$$
$$Q_2 = -10\ V\ cos\theta + 10\ V^2$$

Calculate P_1 and Q_1 once we know V and θ

We know P_2 and Q_2 → Solve for V and θ (implicit equations)

Solving the implicit equations

$$\begin{cases} P_2 = 10\,V\,sin\theta \\ Q_2 = -10\,V\,cos\theta + 10\,V^2 \end{cases}$$

$$P_2 = -0.5\,pu$$
$$Q_2 = -0.5\,pu$$

Rewrite these as follows:

$$\begin{cases} P(\theta,V) \equiv 10\,V\,sin\theta + 0.5 = 0 \\ Q(\theta,V) \equiv 10\,V^2 - 10V\,cos\theta + 0.5 = 0 \end{cases}$$

$$\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases}$$

$P(\theta,V)$ and $Q(\theta,V)$ are called respectively the active and reactive mismatches because they represent the difference between the active and reactive power injected at a node and the active and reactive power flowing into this node through the branches.

Jacobian of the power flow equations

$$\begin{cases} P(\theta,V) \equiv 10\,V\sin\theta + 0.5 = 0 \\ Q(\theta,V) \equiv 10\,V^2 - 10\,V\cos\theta + 0.5 = 0 \end{cases}$$

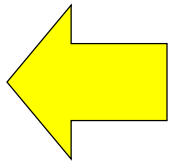
$$J = \begin{pmatrix} \dfrac{\partial P(\theta,V)}{\partial \theta} & \dfrac{\partial P(\theta,V)}{\partial V} \\ \dfrac{\partial Q(\theta,V)}{\partial \theta} & \dfrac{\partial Q(\theta,V)}{\partial V} \end{pmatrix} = \begin{pmatrix} 10\,V\cos\theta & 10\sin\theta \\ 10\,V\sin\theta & 20\,V - 10\cos\theta \end{pmatrix}$$

$$J\begin{pmatrix} \Delta\theta \\ \Delta V \end{pmatrix} = -\begin{pmatrix} P(\theta^0,V^0) \\ Q(\theta^0,V^0) \end{pmatrix}$$

$$\begin{pmatrix} \Delta\theta \\ \Delta V \end{pmatrix} = -\,J^{-1}\begin{pmatrix} P(\theta^0,V^0) \\ Q(\theta^0,V^0) \end{pmatrix}$$

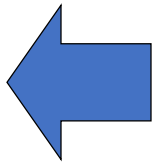
First Iteration

$$\begin{cases} q^0 = 0.0 \\ V^0 = 1.0 \end{cases}$$

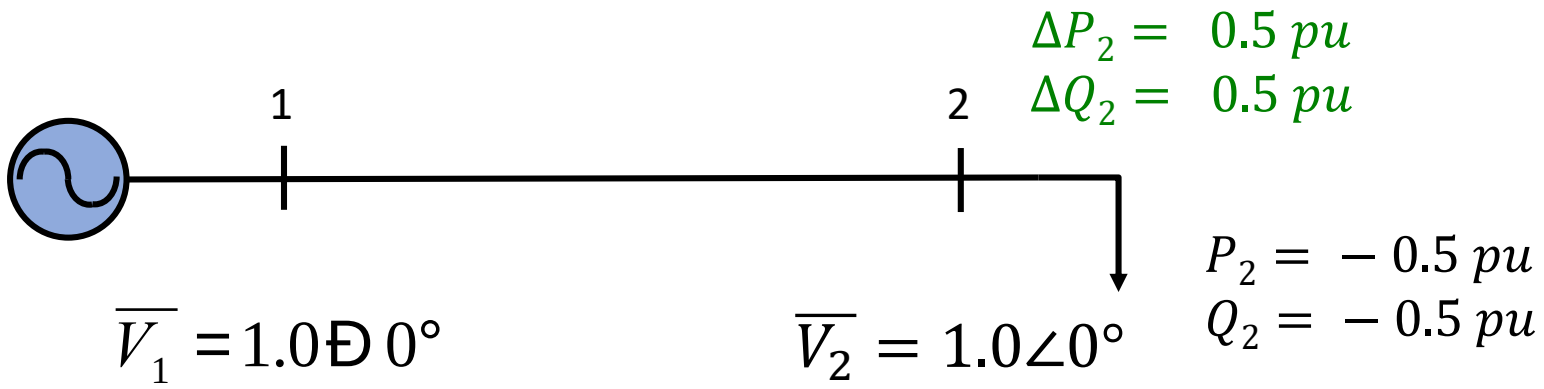


"Flat Start"

$$\begin{cases} P(q^0, V^0) \equiv 10 V^0 \sin q^0 + 0.5 = 0.5 \\ Q(q^0, V^0) \equiv 10 V^{0\,2} - 10 V^0 \cos q^0 + 0.5 = 0.5 \end{cases}$$

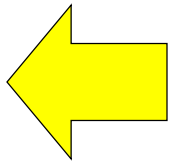


Non-zero mismatches



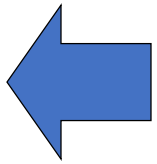
First Iteration

$$\begin{cases} q^0 = 0.0 \\ V^0 = 1.0 \end{cases}$$



"Flat Start"

$$\begin{cases} P(q^0, V^0) \equiv 10 V^0 \sin q^0 + 0.5 = 0.5 \\ Q(q^0, V^0) \equiv 10 V^{0\,2} - 10 V^0 \cos q^0 + 0.5 = 0.5 \end{cases}$$



Non-zero mismatches

$$J = \begin{pmatrix} 10 V^0 \cos q^0 & 10 \sin q^0 \\ 10 V^0 \sin q^0 & 20 V^0 - 10 \cos q^0 \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} \qquad J^{-1} = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}$$

$$\begin{pmatrix} \Delta q \\ \Delta V \end{pmatrix} = - J^{-1} \begin{pmatrix} P(q^0, V^0) \\ Q(q^0, V^0) \end{pmatrix} = \begin{pmatrix} -0.05 \\ -0.05 \end{pmatrix}$$



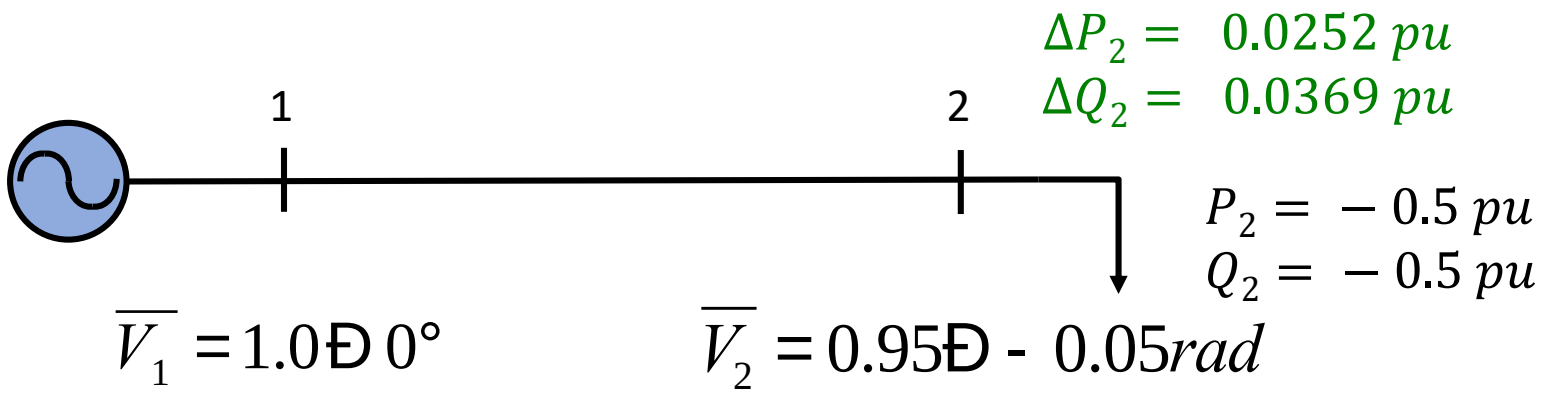
$$\begin{pmatrix} q^1 \\ V^1 \end{pmatrix} = \begin{pmatrix} q^0 \\ V^0 \end{pmatrix} + \begin{pmatrix} \Delta q \\ \Delta V \end{pmatrix} = \begin{pmatrix} -0.05 \\ 0.95 \end{pmatrix}$$

Second Iteration

$$\begin{cases} q^1 = -0.05 \\ V^1 = 0.95 \end{cases}$$

Reduced mismatches

$$\begin{cases} P(q^1, V^1) \equiv 10 V^1 \sin q^1 + 0.5 = 0.0252 \\ Q(q^1, V^1) \equiv 10 (V^1)^2 - 10 V^1 \cos q^1 + 0.5 = 0.0369 \end{cases}$$



Second Iteration

$$\begin{cases} q^1 = -0.05 \\ V^1 = 0.95 \end{cases}$$

$$\begin{cases} P(q^1, V^1) \equiv 10 V^1 \sin q^1 + 0.5 = 0.0252 \\ Q(q^1, V^1) \equiv 10 (V^1)^2 - 10 V^1 \cos q^1 + 0.5 = 0.0369 \end{cases}$$

$$J = \begin{pmatrix} 9.488 & -0.4998 \\ -0.4748 & 9.0125 \end{pmatrix}$$

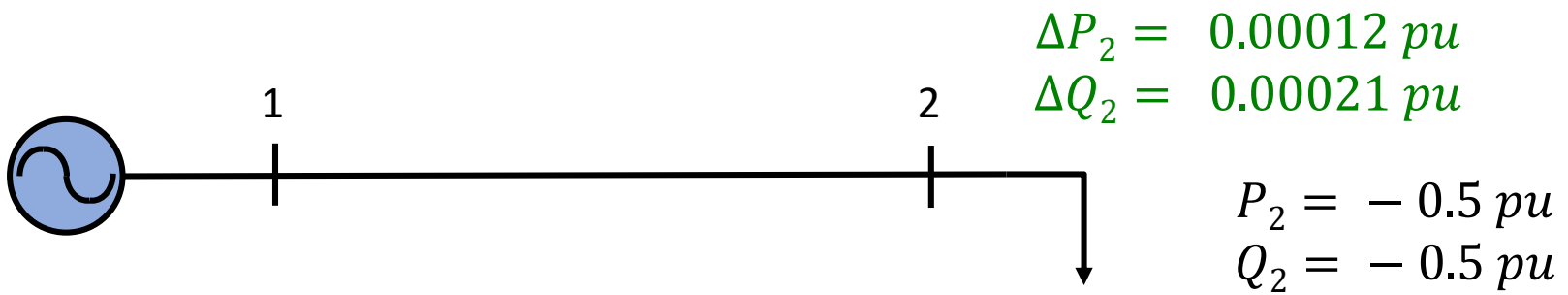
$$\begin{pmatrix} \Delta\theta \\ \Delta V \end{pmatrix} = -J^{-1} \begin{pmatrix} P(\theta^1, V^1) \\ Q(\theta^1, V^1) \end{pmatrix} = \begin{pmatrix} -0.00288 \\ -0.00424 \end{pmatrix} \Rightarrow \begin{pmatrix} \theta^2 \\ V^2 \end{pmatrix} = \begin{pmatrix} \theta^1 \\ V^1 \end{pmatrix} + \begin{pmatrix} \Delta\theta \\ \Delta V \end{pmatrix} = \begin{pmatrix} -0.0529 \\ 0.9458 \end{pmatrix}$$

Third Iteration

$$\begin{cases} \theta^2 = -0.05288 \\ V^2 = 0.9458 \end{cases}$$

$$\begin{cases} P(\theta^2, V^2) \equiv 10V^2 \sin(\theta^2) + 0.5 = 0.00012 \\ Q(\theta^2, V^2) \equiv 10\,(V^2)^2 - 10V^2 \cos(\theta^2) + 0.5 = 0.00021 \end{cases}$$

Mismatches almost small enough to stop



$\Delta P_2 = \ 0.00012\ pu$

$\Delta Q_2 = \ 0.00021\ pu$

$P_2 = \ -0.5\ pu$

$Q_2 = \ -0.5\ pu$

$\overline{V_1} = 1.0 \angle 0°$

$\overline{V_2} = 0.9458 \angle -0.05289\ rad$

Solution (after one more iteration)

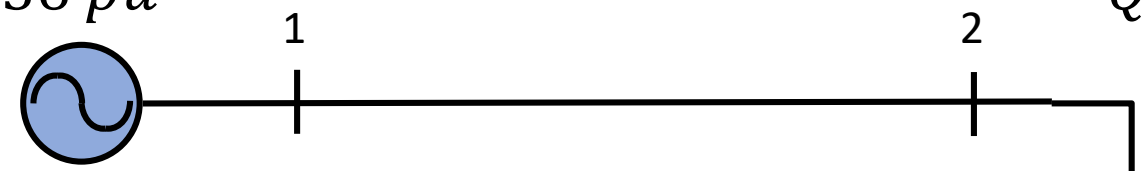
$$\begin{cases} \theta = -0.05289\ rad = -3.03° \\ V = 0.9457\ p.u. \end{cases}$$

Plug θ and V in the explicit equations for the slack bus

$$\begin{cases} P_1 = -10\ V\ sin\theta = 0.5 \\ Q_1 = 10 - 10\ V\ cos\theta = 0.556 \end{cases}$$

$P_1 = 0.5\ pu$
$Q_1 = 0.556\ pu$

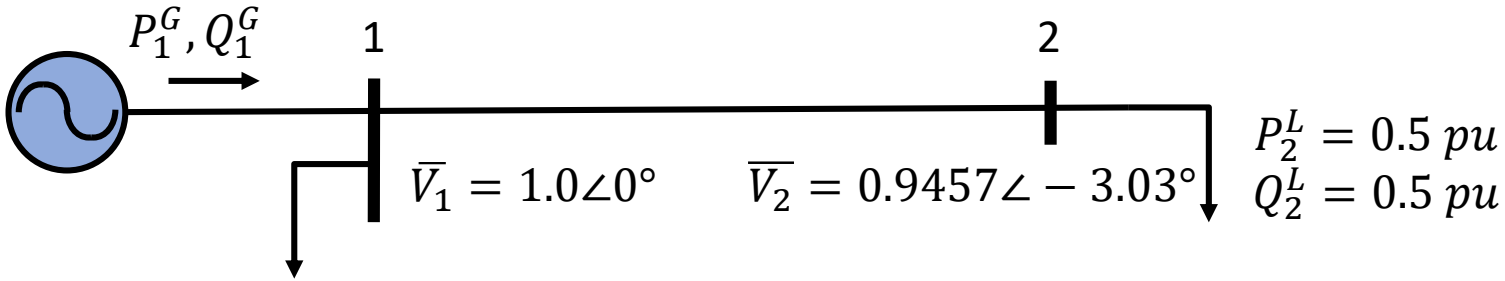
$P_2 = -0.5\ pu$
$Q_2 = -0.5\ pu$



$\overline{V_1} = 1.0 \angle 0°$

$\overline{V_2} = 0.9457 \angle -3.03°$

Example: Two-bus power flow



$P_1^L = 0.2\ pu$
$Q_1^L = 0.1\ pu$

$P_1 = 0.5\ pu$
$Q_1 = 0.556\ pu$

$P_1^G = P_1 + P_1^L = 0.7\ pu$
$Q_1^G = Q_1 + Q_1^L + 0.656\ pu$

Application of the NR method to the power flow

Powers injected at node k

$$P_k = \sum_{i=1}^{N} V_k V_i [G_{ki} cos\theta_{ki} + B_{ki} sin\theta_{ki}]$$

$$Q_k = \sum_{i=1}^{N} V_k V_i [G_{ki} sin\theta_{ki} - B_{ki} cos\theta_{ki}]$$

Powers flowing on the lines connected to node k

$$\sum_{i=1}^{N} V_k V_i [G_{ki} cos\theta_{ki} + B_{ki} sin\theta_{ki}] - P_k = 0$$

$$\sum_{i=1}^{N} V_k V_i [G_{ki} sin\theta_{ki} - B_{ki} cos\theta_{ki}] - Q_k = 0$$

$$P_k = P_k^G - P_k^L$$

$$Q_k = Q_k^G - Q_k^L$$

$$k = 1, \dots N$$

$$\begin{cases} P(\theta_1, \dots, \theta_N, V_1, \dots, V_N,) = 0 \\ Q(\theta_1, \dots, \theta_N, V_1, \dots, V_N,) = 0 \end{cases}$$

2N mismatch equations

Generalization to N busses

- System with N buses → 2N voltage magnitudes and angles:
  - $(\theta_1, \theta_2, \dots, \theta_N, V_1, V_2, \dots, V_N)$
- Suppose bus 1 is the slack bus
  - θ_1=0
  - V_1 is given and thus known
- Assume that buses 2 to m are PV buses
  - $V_2 \dots V_m$ are given and thus known
- There are then $N - 1 - m$ PQ buses
- $N - 1$ unknown voltage angles: $\theta_2, \dots, \theta_N$
- $N - 1 - m$ unknown voltage magnitudes: $V_{m+1}, \dots, V_N$

Newton-Raphson algorithm for power flow

1. Initialize
    1. Read input data
    2. Build the admittance matrix
    3. Choose starting point for voltages and angles
2. Until mismatches are sufficiently small, solve the implicit equations:
    1. Calculate the mismatches
    2. Build the Jacobian matrix
    3. Invert the Jacobian matrix
    4. Calculate the corrections
    5. Update the voltage magnitudes and angles
3. Using the explicit equations, calculate the active injection at the slack bus and the reactive injections at the PV buses
4. Calculate the line flows

Implicit equations in mismatch form

$$
\begin{cases}
P_2(\theta_2, \dots, \theta_N, V_{m+1}, \dots, V_N) = 0 \\
\dots \\
P_N(\theta_2, \dots, \theta_N, V_{m+1}, \dots, V_N) = 0 \\
Q_{m+1}(\theta_2, \dots, \theta_N, V_{m+1}, \dots, V_N) = 0 \\
\dots \\
Q_N(\theta_2, \dots, \theta_N, V_{m+1}, \dots, V_N) = 0
\end{cases}
$$

Active power balance equations at the PQ and PV buses

Reactive power balance equations at the PQ buses

$2N - 1 - m$ equations

$2N - 1 - m$ variables

Calculate the mismatches

$P_2(\theta_2, \dots, \theta_N, V_{m+1}, \dots, V_N)$

…

$P_N(\theta_2, \dots, \theta_N, V_{m+1}, \dots, V_N)$

$Q_{m+1}(\theta_2, \dots, \theta_N, V_{m+1}, \dots, V_N)$

…

$Q_N(\theta_2, \dots, \theta_N, V_{m+1}, \dots, V_N)$

Build the Jacobian

$$\begin{cases} P_2(\theta_2, \dots, \theta_N, V_{m+1}, \dots, V_N) = 0 \\ \dots \\ P_N(\theta_2, \dots, \theta_N, V_{m+1}, \dots, V_N) = 0 \\ Q_{m+1}(\theta_2, \dots, \theta_N, V_{m+1}, \dots, V_N) = 0 \\ \dots \\ Q_N(\theta_2, \dots, \theta_N, V_{m+1}, \dots, V_N) = 0 \end{cases}$$

$$J = \begin{bmatrix} \frac{\partial P_2}{\partial \theta_2} & \cdots & \frac{\partial P_2}{\partial \theta_N} & \frac{\partial P_2}{\partial V_{m+1}} & \cdots & \frac{\partial P_2}{\partial V_N} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial P_N}{\partial \theta_2} & \cdots & \frac{\partial P_N}{\partial \theta_N} & \frac{\partial P_N}{\partial V_{m+1}} & \cdots & \frac{\partial P_N}{\partial V_N} \\ \frac{\partial Q_{m+1}}{\partial \theta_2} & \cdots & \frac{\partial Q_{m+1}}{\partial \theta_N} & \frac{\partial Q_{m+1}}{\partial V_{m+1}} & \cdots & \frac{\partial Q_{m+1}}{\partial V_N} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial Q_N}{\partial \theta_2} & \cdots & \frac{\partial Q_N}{\partial \theta_N} & \frac{\partial Q_N}{\partial V_{m+1}} & \cdots & \frac{\partial Q_N}{\partial V_N} \end{bmatrix}$$

Invert the Jacobian and calculate corrections

$$\begin{bmatrix} \Delta\theta_2 \\ \dots \\ \Delta\theta_N \\ \\ \Delta V_{m+1} \\ \dots \\ \Delta V_N \end{bmatrix} = -J^{-1} \begin{bmatrix} P_2(\theta_2, \dots, \theta_N, V_{m+1}, \dots, V_N) \\ \dots \\ P_N(\theta_2, \dots, \theta_N, V_{m+1}, \dots, V_N) \\ \\ Q_{m+1}(\theta_2, \dots, \theta_N, V_{m+1}, \dots, V_N) \\ \dots \\ Q_N(\theta_2, \dots, \theta_N, V_{m+1}, \dots, V_N) \end{bmatrix}$$

Update the voltage magnitudes and angles

$$\begin{bmatrix} \theta_2^{k+1} \\ \dots \\ \theta_N^{k+1} \\ V_{m+1}^{k+1} \\ \dots \\ V_N^{k+1} \end{bmatrix} = \begin{bmatrix} \theta_2^{k} \\ \dots \\ \theta_N^{k} \\ V_{m+1}^{k} \\ \dots \\ V_N^{k} \end{bmatrix} + \begin{bmatrix} \Delta\theta_2 \\ \dots \\ \Delta\theta_N \\ \Delta V_{m+1} \\ \dots \\ \Delta V_N \end{bmatrix}$$

Terms of the Jacobian

$$J = \left[\begin{array}{ccc|ccc} \frac{\partial P_2}{\partial \theta_2} & \cdots & \frac{\partial P_2}{\partial \theta_N} & \frac{\partial P_2}{\partial V_{m+1}} & \cdots & \frac{\partial P_2}{\partial V_N} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial P_N}{\partial \theta_2} & \cdots & \frac{\partial P_N}{\partial \theta_N} & \frac{\partial P_N}{\partial V_{m+1}} & \cdots & \frac{\partial P_N}{\partial V_N} \\ \hline \frac{\partial Q_{m+1}}{\partial \theta_2} & \cdots & \frac{\partial Q_{m+1}}{\partial \theta_N} & \frac{\partial Q_{m+1}}{\partial V_{m+1}} & \cdots & \frac{\partial Q_{m+1}}{\partial V_N} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial Q_N}{\partial \theta_2} & \cdots & \frac{\partial Q_N}{\partial \theta_N} & \frac{\partial Q_N}{\partial V_{m+1}} & \cdots & \frac{\partial Q_N}{\partial V_N} \end{array}\right] = \begin{pmatrix} H & M \\ N & L \end{pmatrix}$$

H terms of the Jacobian matrix

$$P_k(\theta_2, \dots, \theta_N, V_{m+1}, \dots, V_N) \equiv \sum_{i=1}^{N} V_k V_i [G_{ki} cos(\theta_k - \theta_i) + B_{ki} sin(\theta_k - \theta_i)] - P_k$$

For $i \neq k$: $H_{ki} = \frac{\partial P_k}{\partial \theta_i} = V_k V_i [G_{ki} \sin(\theta_k - \theta_i) - B_{ki} cos(\theta_k - \theta_i)]$

For $i = k$: $H_{kk} = \frac{\partial P_k}{\partial \theta_k} = \sum_{\substack{i=1 \\ i \neq k}}^{N} V_k V_i [-G_{ki} sin(\theta_k - \theta_i) + B_{ki} cos(\theta_k - \theta_i)]$

M terms of the Jacobian matrix

$$P_k(\theta_2, \dots, \theta_N, V_{m+1}, \dots, V_N) \equiv \sum_{i=1}^{N} V_k V_i [G_{ki} cos(\theta_k - \theta_i) + B_{ki} sin(\theta_k - \theta_i)] - P_k$$

For $i \neq k$: $M_{ki} = \frac{\partial P_k}{\partial V_i} = V_k [G_{ki} cos(\theta_k - \theta_i) + B_{ki} sin(\theta_k - \theta_i)]$

For $i = k$: $M_{kk} = \frac{\partial P_k}{\partial V_k} = 2G_{kk} V_k + \sum_{\substack{i=1 \\ i \neq k}}^{N} V_i [G_{ki} cos(\theta_k - \theta_i) + B_{ki} sin(\theta_k - \theta_i)]$

N terms of the Jacobian matrix

$$Q_k(\theta_2, \dots, \theta_N, V_{m+1}, \dots, V_N) = \sum_{i=1}^{N} V_k V_i [G_{ki} sin(\theta_k - \theta_i) - B_{ki} cos(\theta_k - \theta_i)] - Q_k = 0$$

For $i \neq k$: $N_{ki} = \frac{\partial Q_k}{\partial \theta_i} = V_k V_i [-G_{ki} cos(\theta_k - \theta_i) - B_{ki} sin(\theta_k - \theta_i)]$

For $i = k$: $N_{kk} = \frac{\partial Q_k}{\partial \theta_k} = \sum_{\substack{i=1 \\ i \neq k}}^{N} V_k V_i [G_{ki} cos(\theta_k - \theta_i) + B_{ki} sin(\theta_k - \theta_i)]$

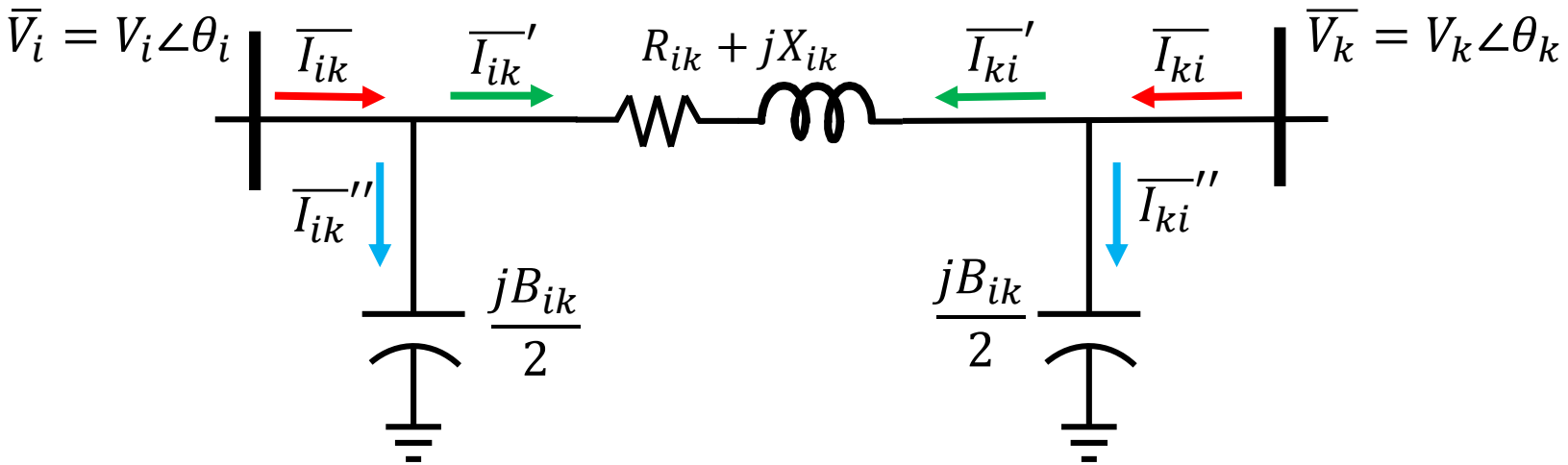
L terms of the Jacobian matrix

$$Q_k(\theta_2, \dots, \theta_N, V_{m+1}, \dots, V_N) = \sum_{i=1}^{N} V_k V_i [G_{ki} sin(\theta_k - \theta_i) - B_{ki} cos(\theta_k - \theta_i)] - Q_k$$

For $i \neq k$: $L_{ki} = \frac{\partial Q_k}{\partial V_i} = V_k [G_{ki} sin(\theta_k - \theta_i) - B_{ki} cos(\theta_k - \theta_i)]$

For $i = k$: $L_{kk} = \frac{\partial Q_k}{\partial V_k} = -2B_{kk}V_k + \sum_{\substack{i=1 \\ i \neq k}}^{N} V_i [G_{ki} sin(\theta_k - \theta_i) - B_{ki} cos(\theta_k - \theta_i)]$

Calculating the lines flows



$$\overline{I_{ik}} = \overline{I_{ik}}' + \overline{I_{ik}}''$$

$$\overline{I_{ik}}' = \frac{\overline{V_i} - \overline{V_k}}{R_{ik} + jX_{ik}}$$

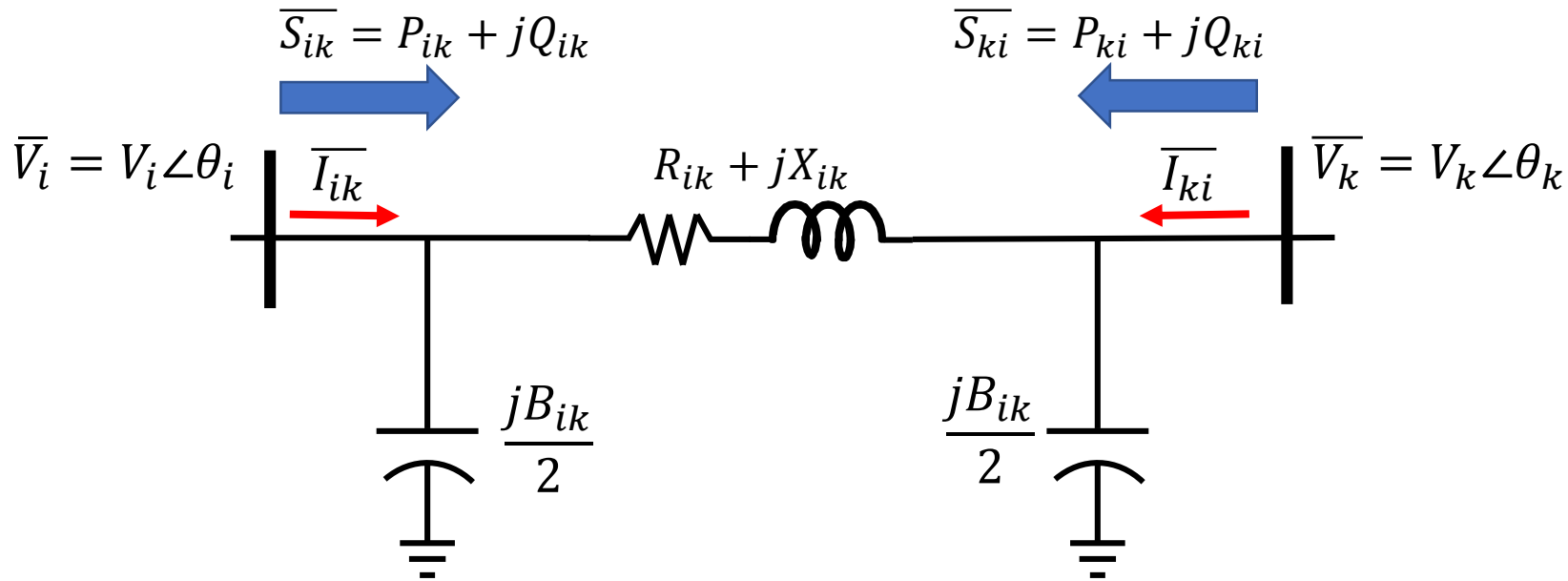
$$\overline{I_{ik}}'' = \frac{jB_{ik}}{2}\overline{V_i}$$

$$\overline{I_{ki}} = \overline{I_{ki}}' + \overline{I_{ki}}''$$

$$\overline{I_{ki}}' = \frac{\overline{V_k} - \overline{V_i}}{R_{ik} + jX_{ik}}$$

$$\overline{I_{ki}}'' = \frac{jB_{ik}}{2}\overline{V_k}$$

Calculating the lines flows



$$\overline{S_{ik}} = P_{ik} + jQ_{ik} = \overline{V_i}\,\overline{I_{ik}}^{*}$$

$$\overline{S_{ki}} = P_{ki} + jQ_{ki} = \overline{V_k I_{ki}}^{*}$$

$\overline{S_{ik}} \neq -\overline{S_{ki}}$ because of the active and reactive losses

Power Flow Applications

Basic power flow functions

- Calculate bus voltage magnitude and angle
- Calculate line flows

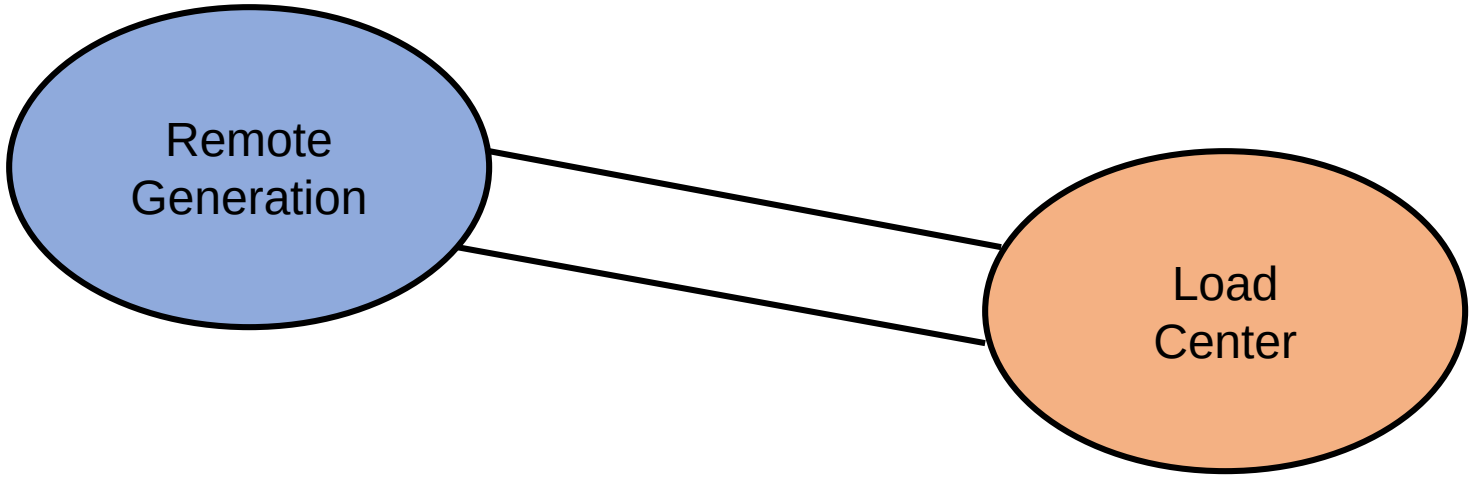
- Check voltages and flows against limits for all probable situations
- Initialize other calculations (dynamics, faults)

Applications in Planning

- Look a few months to a few years ahead
- Forecast the load at all busses
- Forecast construction and retirement of generating plants
- Calculate voltages and flows for various generation and load scenarios
  - Peak load
  - Light load
  - Various lines and generating plants on outage
- Are all voltages and flows OK under all these conditions?

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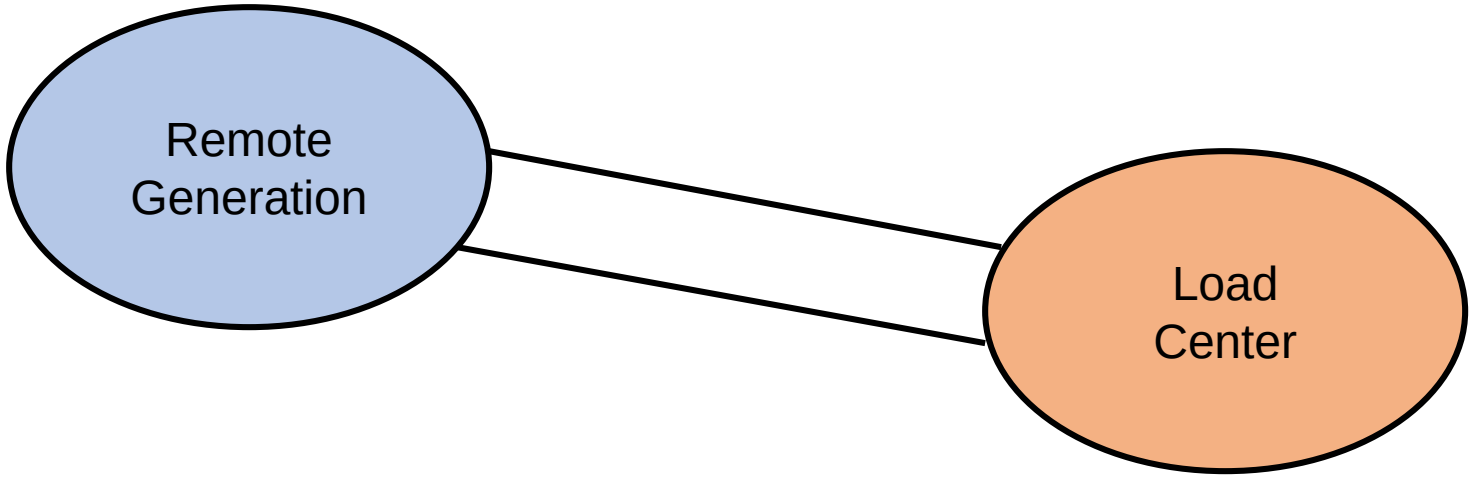
What if flows are too high?



- Build more lines
- Upgrade lines to higher nominal voltage
- Build generating plants closer to the load center
- Encourage energy efficiency to reduce load

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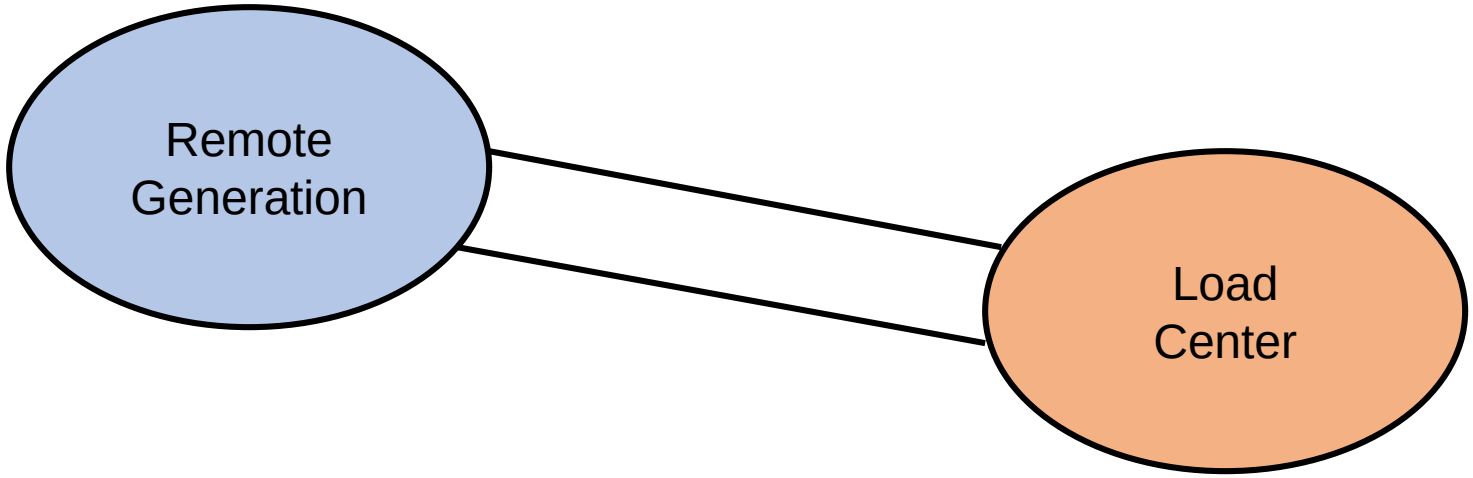
What if voltages are too low?



- Build more lines
- Upgrade lines to higher nominal voltage
- Build plants closer to the load center
- Encourage energy efficiency to reduce load
- Add sources of reactive power closer to load

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What if voltages are too high?



- Add sinks of reactive power close to the load center

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Applications in Operation

- Look a few hours to a few months ahead
- Adding new plants or lines is not an option
- Must operate the existing system at minimum cost while maintaining system reliability
  - Check normal conditions
    - All equipment (except those on maintenance) are in service
  - Contingency analysis

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Contingency analysis

- Contingency: an event that might happen
  - Sudden generator outage
  - Sudden line or transformer outage
- Credible contingency: an event that has a non-negligible probability of happening
  - N-1 contingencies (loss of a single component) are much more likely than N-2 contingencies
- Contingency analysis: perform a power flow for each credible contingency and check that the resulting state is still OK

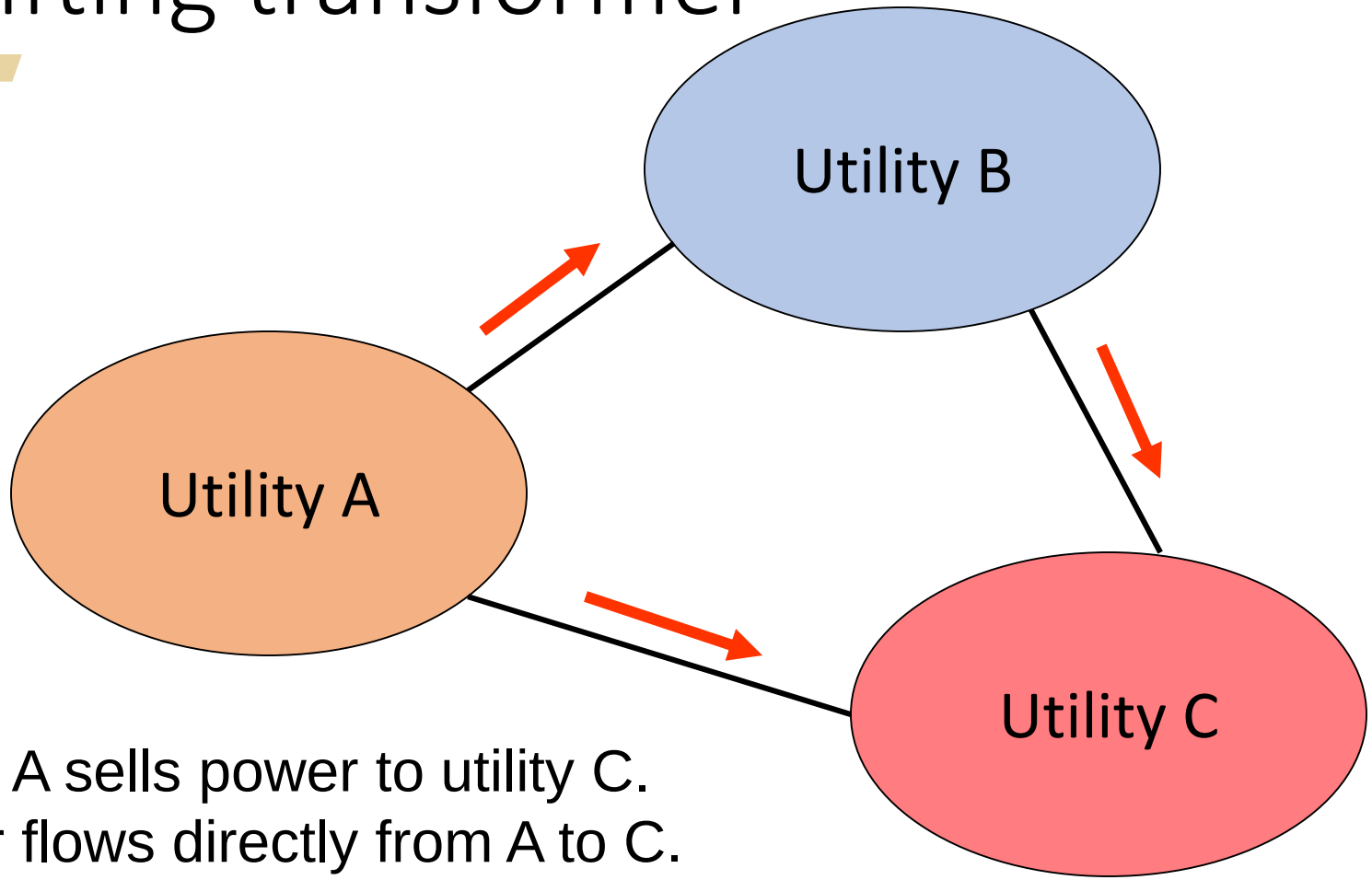
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Active Power Flow Control

- Used to correct excessive flow in lines
- Generation re-dispatch
  - Very effective but costly
- Phase shifting transformers
  - Introduce an artificial phase angle difference
  - Act as “MW pumps”
- Load shedding
  - Used only as a last resort to avoid a system collapse

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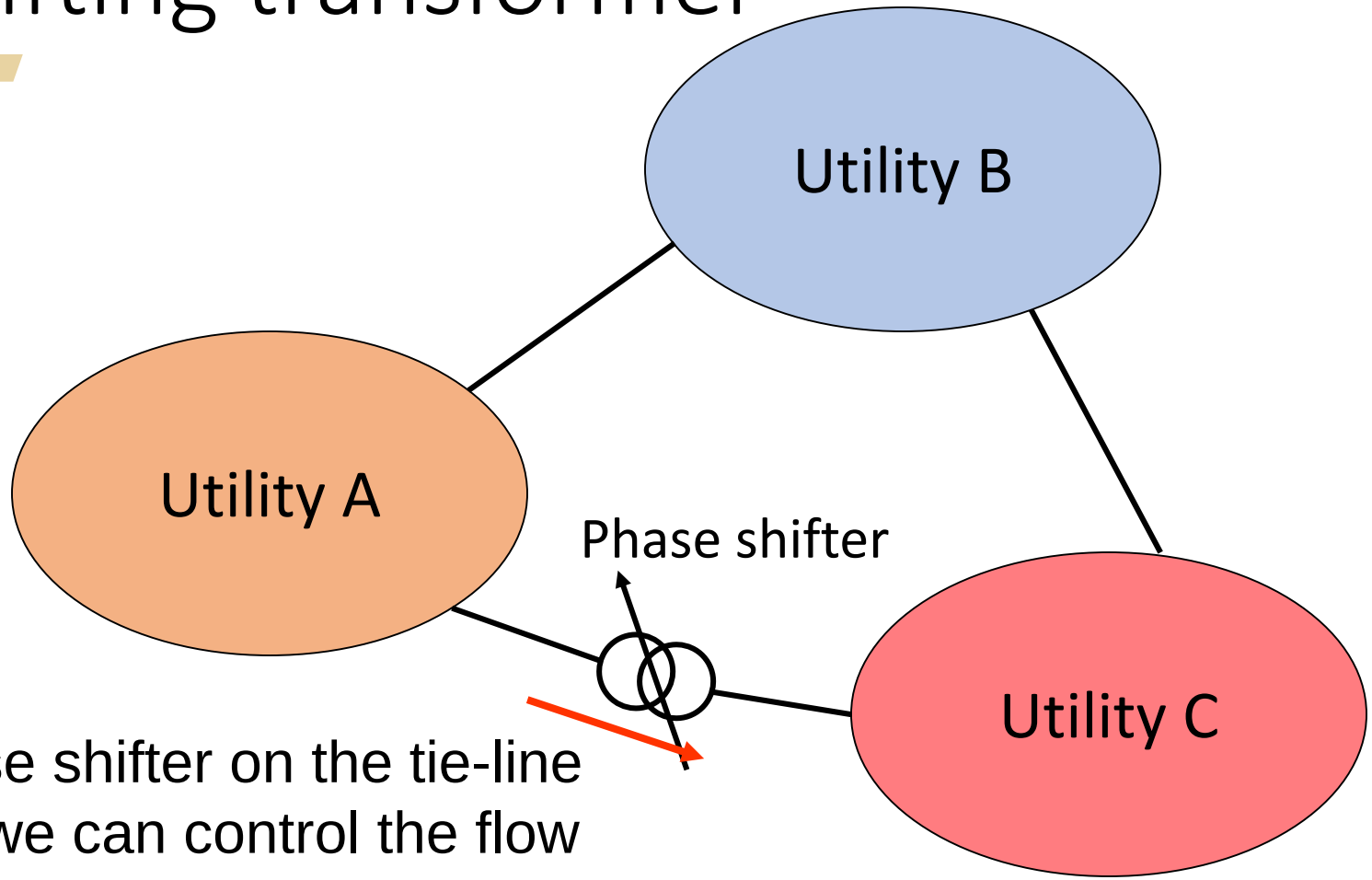
Phase shifting transformer



Suppose that utility A sells power to utility C.
Some of this power flows directly from A to C.
The rest flows from A to C through B.
This so-called “loop flow” may be undesirable

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Phase shifting transformer



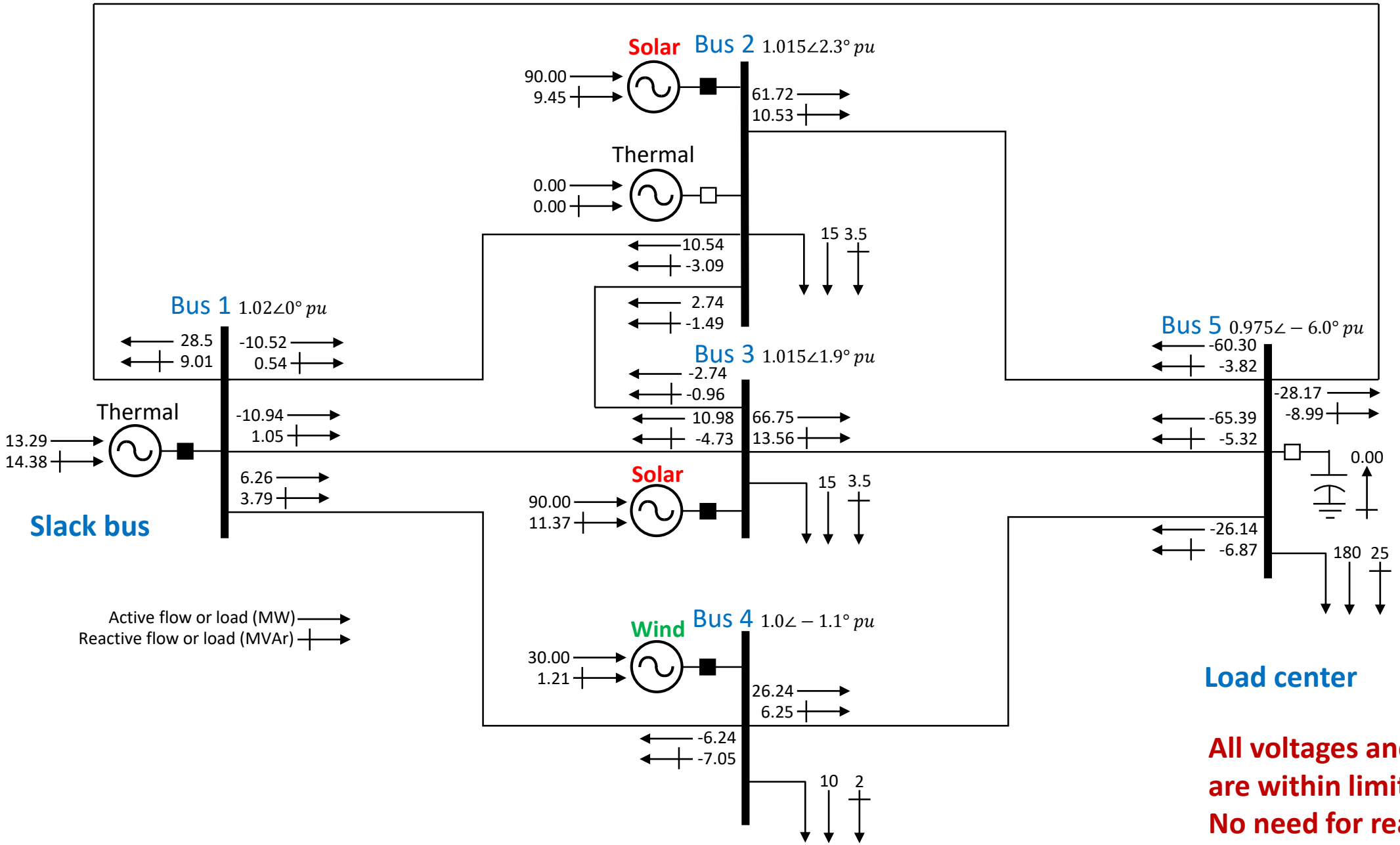
By inserting a phase shifter on the tie-line between A and C, we can control the flow between these two utilities and ensure that the power flows directly between A and C and not through B.

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Reactive Power Flow Control

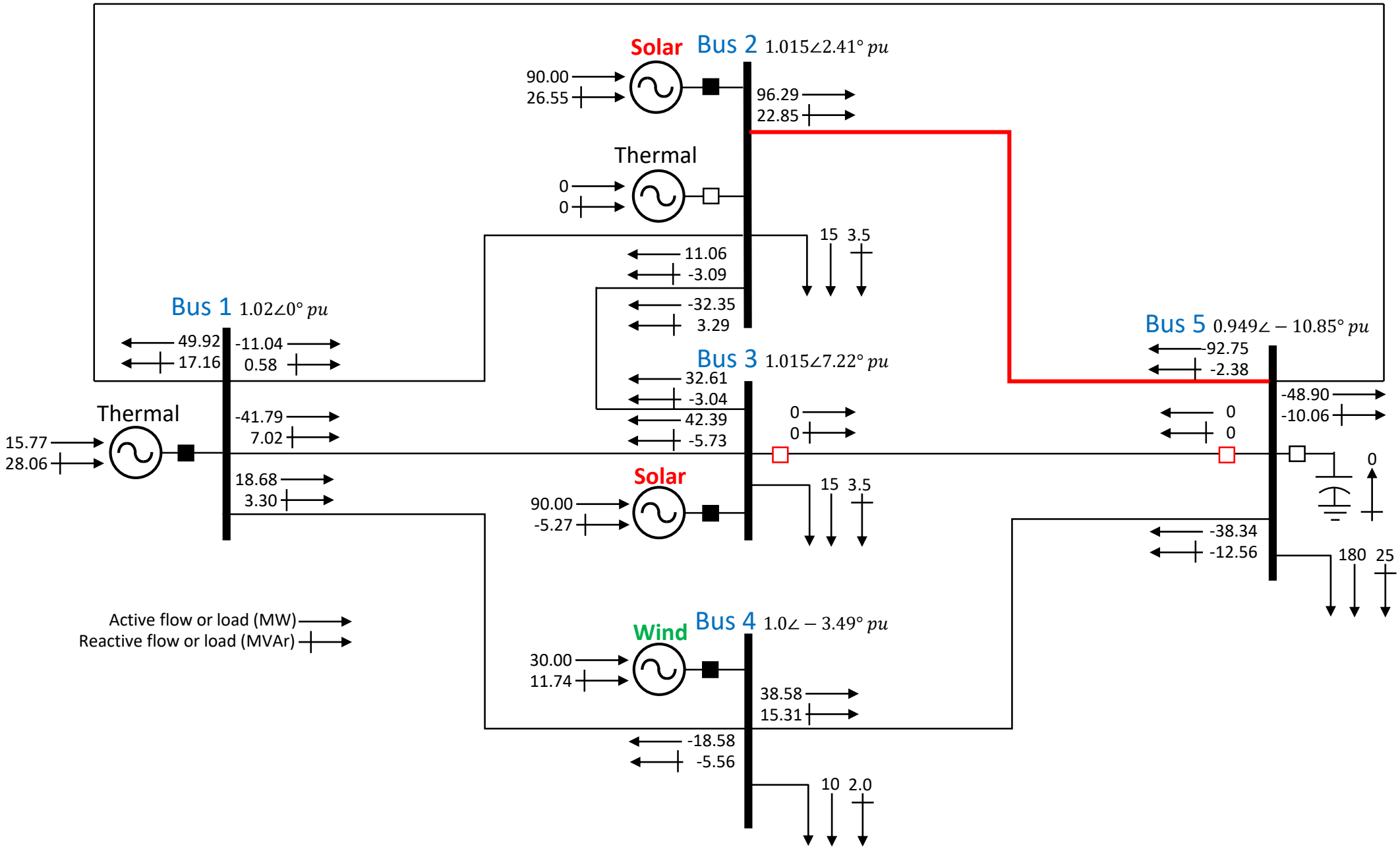
- Used to correct voltage magnitude problems
- Voltage set-point of generators
- Tap-changing transformers
  - Operate like “VAr pumps”
- Reactive compensation
  - Capacitors (mechanically or electronically controlled)
  - Inductors (mechanically or electronically controlled)
  - Synchronous condensers
- Switch lines out of service to lower the voltages
  - Remove shunt capacitance
- Load shedding
  - Used only as a last resort to avoid a system collapse

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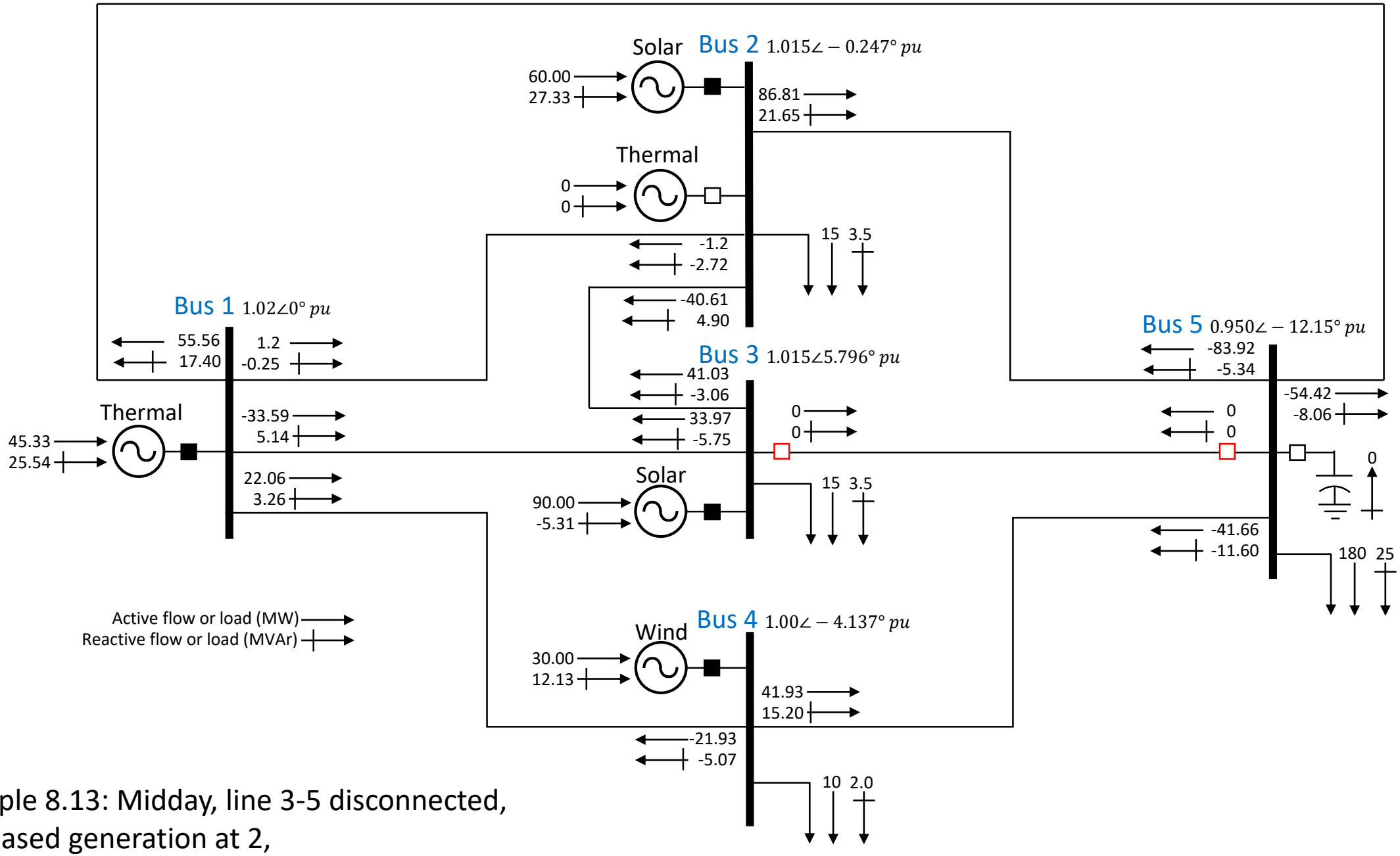
Example 8-11: Midday all lines in service

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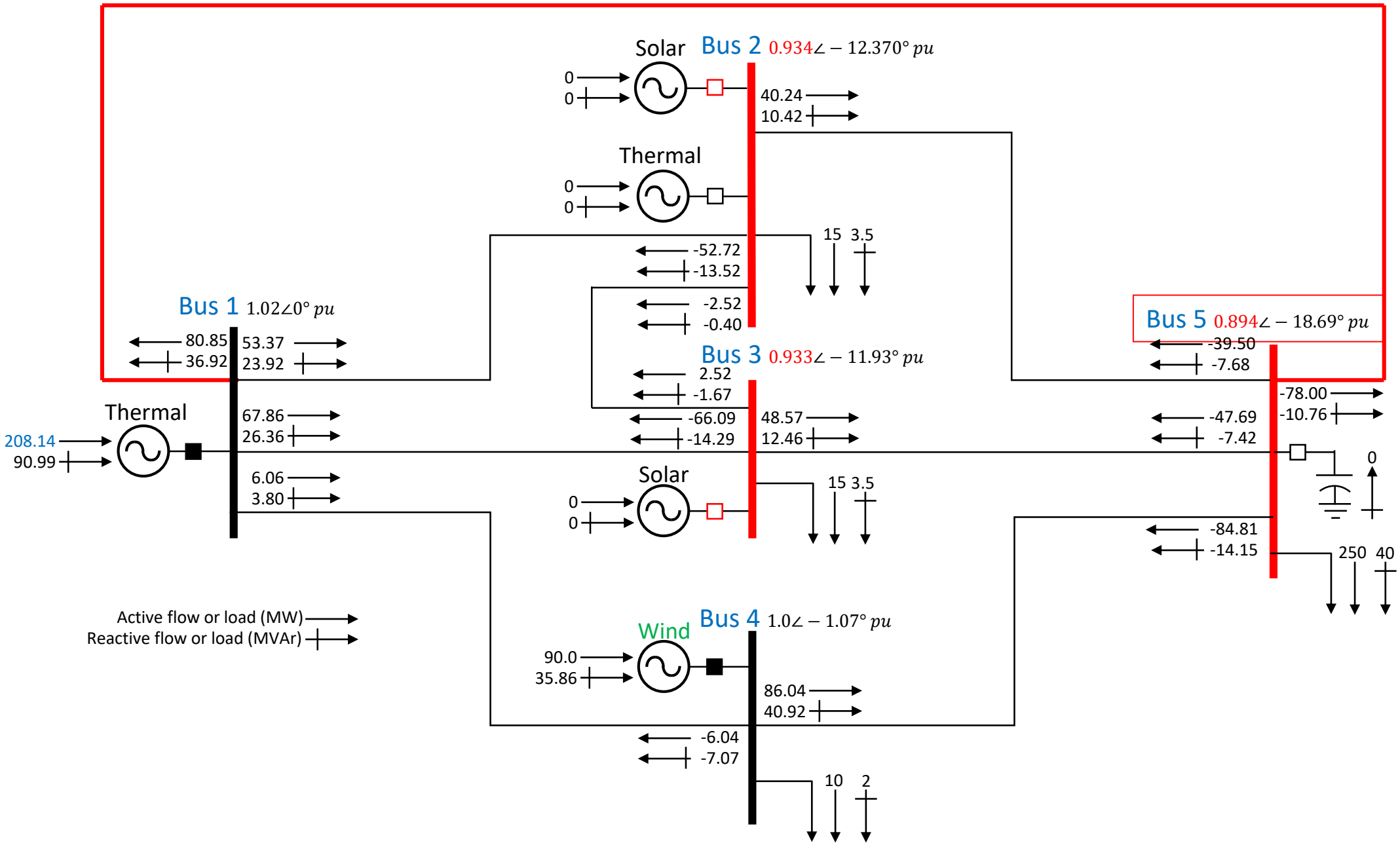
Example 8-12: Midday, line 3-5 disconnected, overload on line 2-5

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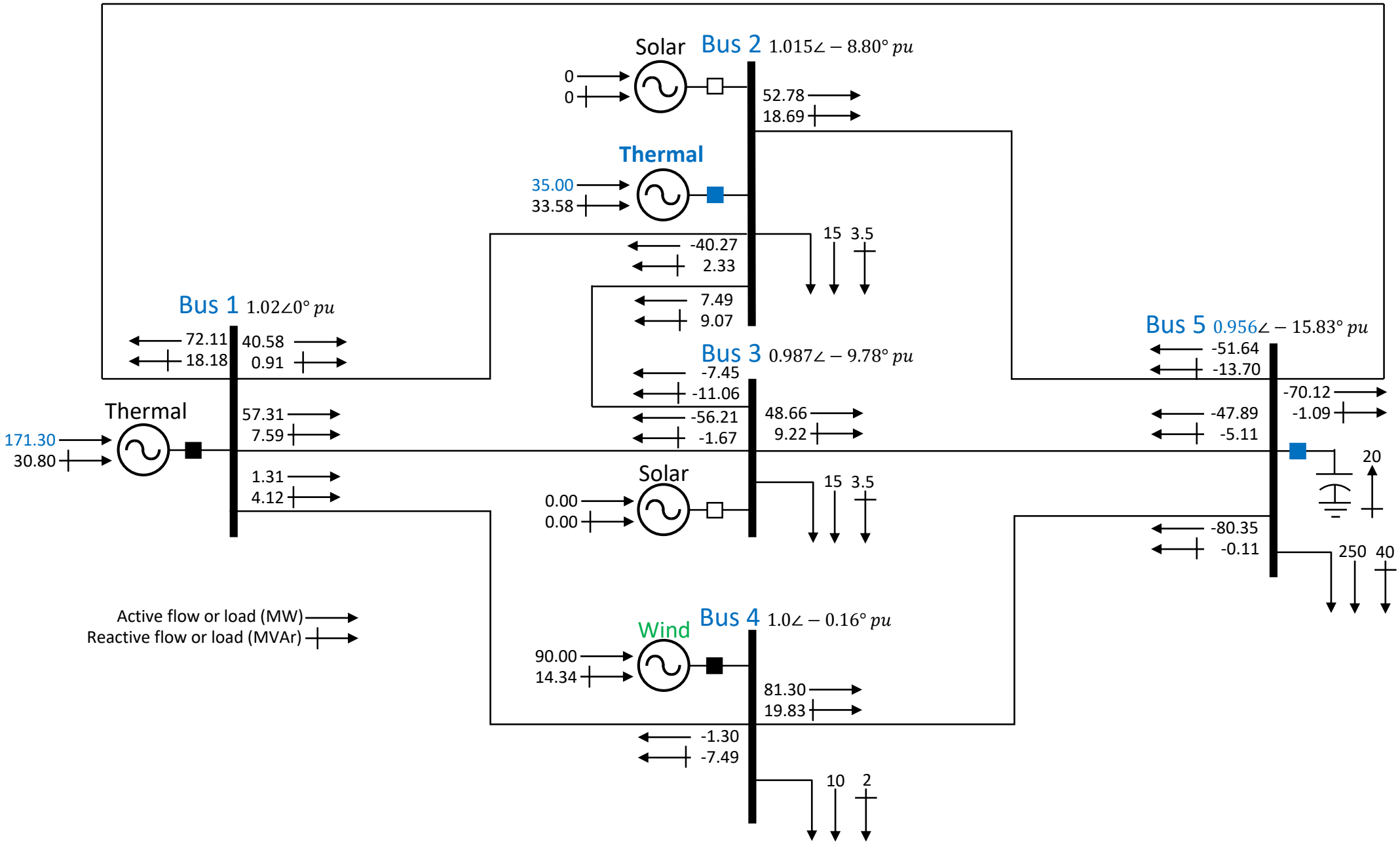
Example 8.13: Midday, line 3-5 disconnected,
decreased generation at 2,
and increased generation at 1 to remove overload on line 2-5

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Example 8.14: Evening peak without corrective action

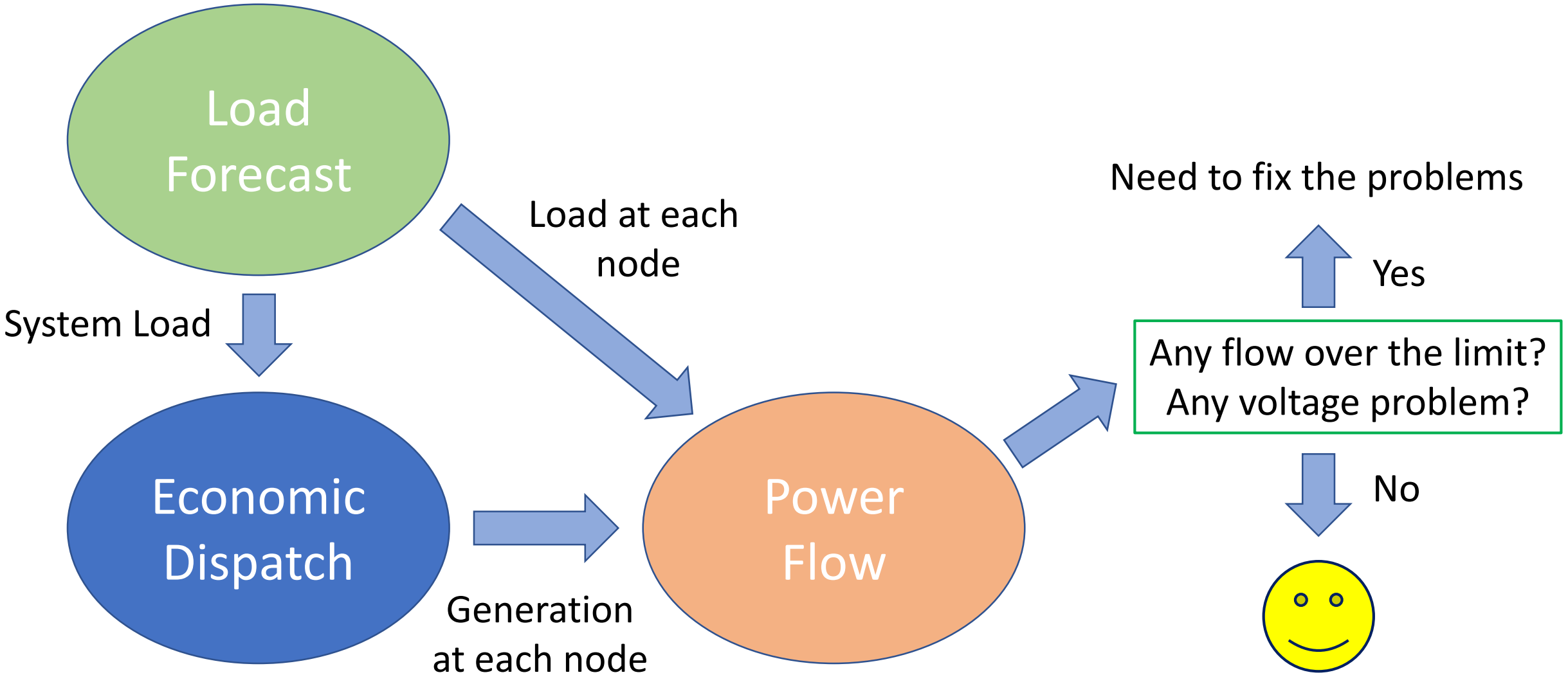
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8.15: Evening peak with corrective actions

Optimal Power Flow

Power System Operation



How do we fix problems?

- How do we fix a problem at minimum cost?
- How do we fix a problem without creating another problem?
  - Example: Removing an overload on one line without overloading another
- How do we fix all the problems at the same time at minimum cost?
- Problem = violation of an inequality constraint
- Formulation as an optimization problem
- Optimal Power Flow (OPF)
  - What values of the control variables minimizes the cost of operating the power system while satisfying the network constraints?

Decision or control variables

- Active power output of the generating units
- Voltage at the generating units
- Position of the transformer taps
- Position of the phase shifter taps
- Status of the switched capacitors and reactors
- Set-point of power electronics (HVDC, SVC)
- Topology of the network
- Amount of load disconnected (as a last resort)

→ Vector of control variables: u

State variables

- Describe the response of the system to changes in the control variables
- Magnitude of voltage at each bus
  - Except generator busses, which are control variables
- Angle of voltage at each bus
  - Except slack bus

→ Vector of state variables: x

Parameters

- Known characteristics of the system
- Network topology
- Network parameters
  - R, X, B
  - Flow and voltage limits
- Load at each bus
- Generator cost functions
- Generator limits

→Vector of parameters: y

Objective function

$$\min_{u} f(u, x, y)$$

- Examples:
  - Minimize generation cost: $\min_{u} \sum_{i \in G} C_i(P_i)$
  - Minimize change in control variables: $\min_{u} \sum_{i \in U} c_i \left| u_i - u_i^0 \right|$

Equality constraints

- Power flow equations:

$$\begin{cases} P_k = \sum_{i=1}^{N} V_k V_i [G_{ki} cos\theta_{ki} + B_{ki} sin\theta_{ki}] \\ Q_k = \sum_{i=1}^{N} V_k V_i [G_{ki} sin\theta_{ki} - B_{ki} cos\theta_{ki}] \end{cases}$$

- Compact notation:

$$g(u, x, y) = 0$$

Inequality constraints

- Limits on the control variables: $u_i^{min} \leq u_i \leq u_i^{max};\ \forall\ u \in U$
- Operating limits on flows: $\sqrt{P_{ij}^2 + Q_{ij}^2} \leq S_{ij}^{max};\ \forall i, j$
- Operating limits on voltages: $V_i^{min} \leq V_i \leq V_i^{max}$
- Compact expression: $h(u, x, y) \leq 0$

Compact formulation of the OPF problem

$$\min_{u} f(u, x, y)$$

Subject to:

$$g(u, x, y) = 0$$

$$h(u, x, y) \leq 0$$

Solving the OPF problem

- Very large optimization problem
  - Many control variables
  - Many inequality constraints
    - Which ones are binding?
- Non-linear objective function
- Non-linear constraints
- Requires powerful software optimization packages