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“Digital Filter Design”

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Digital Filter Design

1.FIR Filter Design.	Frequency Sampling Design Method Fourier Transform Design Method
2. IIR Filter Design.	Bilinear Transformation Method Pole Zero Placement Method
3. Application	60 –Hz Hum Eliminator ECG Pulse

FIR Filter Design

Definition: A finite impulse response (***FIR***) filter is specified by

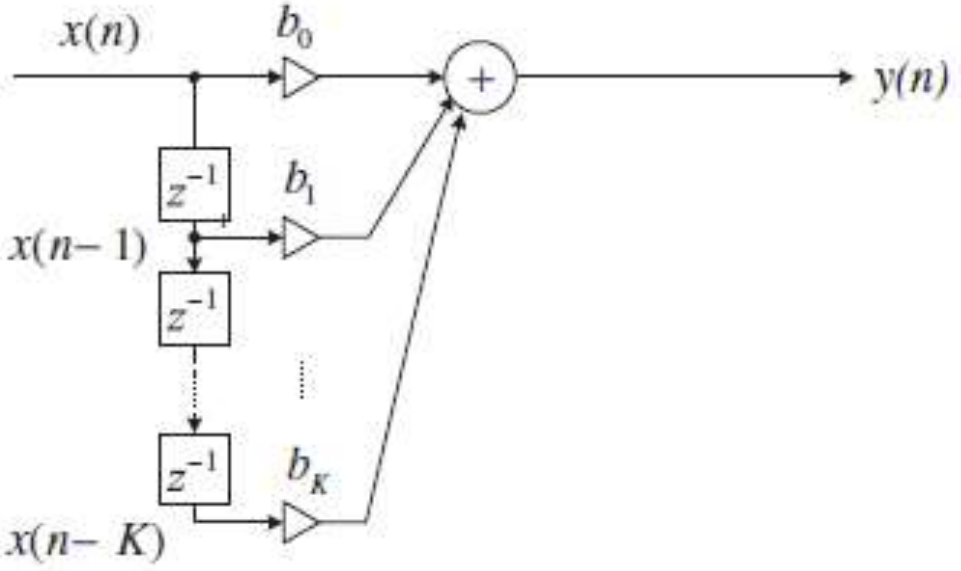
$$y(n) = \sum_{i=0}^{K} b_i x(n-i)$$

The transfer function of the FIR filter: $H(z) = b_0 + b_1 z^{-1} + \cdots + b_K z^{-K}$

b_i filter coefficients, K+1 filter length

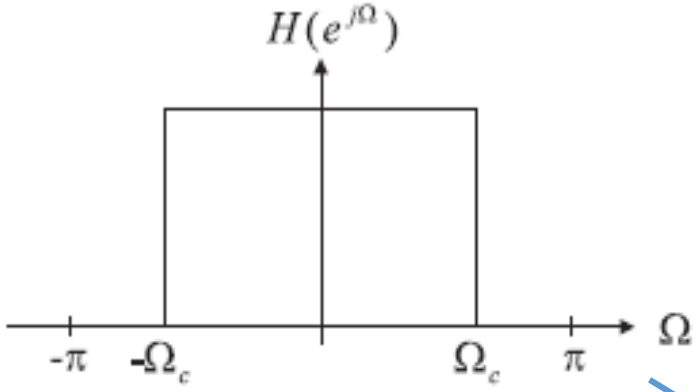
The difference equation $y(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \cdots + b_K x(n-K)$

The realization of the FIR filter



Fourier Transform Design Method

Frequency response of ideal LPF:



$$H(e^{j\Omega}) = \begin{cases} 1, & 0 \le |\Omega| \le \Omega_c \\ 0, & \Omega_c \le |\Omega| \le \pi \end{cases}$$

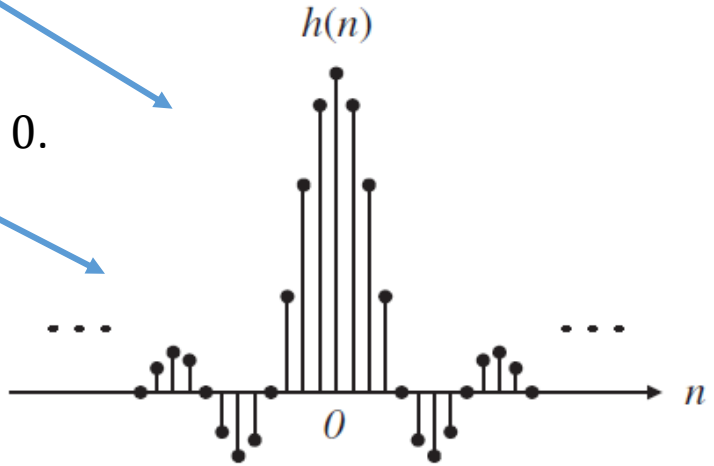
Inverse fourier Transform

Impulse response of ideal LPF:

- The desired impulse response $h(n)$ of ideal LPF is infinite and symmetrical about $n = 0$.
- After truncating $2M+1$ major components using the coefficient symmetry:

$$H(z) = h(M)z^{M} + \cdots + h(1)z^{1} + h(0) + h(1)z^{-1} + \cdots + h(M)z^{-M}$$

Symmetric



- To obtain causal FIR filter, h(n) is delayed by M samples.

$$H(z) = b_0 + b_1 z^{-1} + \cdots + b_{2M} z^{-2M}$$

Where, $b_n = h(n - M)$ for $n = 0, 1, \ldots, 2M$.

Ideal Impulse Responses for Standard FIR Filters

Filter — Type Ideal Impulse Response $h(n)$ (noncausal FIR coefficients)

Lowpass:
$$h(n) = \begin{cases} \dfrac{\Omega_c}{\pi} & \text{for } n = 0 \\ \dfrac{\sin(\Omega_c n)}{n\pi} & \text{for } n \neq 0 \end{cases} \qquad -M \le n \le M$$

Highpass:
$$h(n) = \begin{cases} \dfrac{\pi - \Omega_c}{\pi} & \text{for } n = 0 \\ -\dfrac{\sin(\Omega_c n)}{n\pi} & \text{for } n \neq 0 \end{cases} \qquad -M \le n \le M$$

Bandpass:
$$h(n) = \begin{cases} \dfrac{\Omega_H - \Omega_L}{\pi} & \text{for } n = 0 \\ \dfrac{\sin(\Omega_H n)}{n\pi} - \dfrac{\sin(\Omega_L n)}{n\pi} & \text{for } n \neq 0 \end{cases} \qquad -M \le n \le M$$

Bandstop:
$$h(n) = \begin{cases} \dfrac{\pi - \Omega_H + \Omega_L}{\pi} & \text{for } n = 0 \\ -\dfrac{\sin(\Omega_H n)}{n\pi} + \dfrac{\sin(\Omega_L n)}{n\pi} & \text{for } n \neq 0 \end{cases} \qquad -M \le n \le M$$

FIR Low Pass Filter Design

Example

a. Design a 3-tap FIR LPF with cut-off frequency of 800 Hz and a sampling rate of 8,000 Hz using the Fourier transform method.

b. Determine the transfer function and difference equation of the designed FIR system.

c. Compute and plot the magnitude frequency response for $\Omega = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ and π radians.

Solution

Normalized cut-off frequency ➡ $\Omega_c = 2\pi f_c T_s = 2\pi \times 800/8,000 = 0.2\pi$ radians

3-tap filter ➡ $2M + 1 = 3$ ➡ $M = 1$ ➡ $h(n)\ for\ n\ from\ -M\ to\ M$ ➡ $n = -1, 0, 1,$

filter coefficients ➡ $h(0) = \frac{\Omega_c}{\pi}$ for $n = 0$ and $h(n) = \frac{\sin(\Omega_c n)}{n\pi} = \frac{\sin(0.2\pi n)}{n\pi}$ for $n \neq 1$

From previous slide table

compute coefficients ➡ $h(0) = \frac{0.2\pi}{\pi} = 0.2$ and $h(1) = \frac{\sin[0.2\pi \times 1]}{1 \times \pi} = 0.1871$

Using symmetry ➡ $h(-1) = h(1) = 0.1871$

Delaying h(n) by M = 1 sample ➡ $b_n = h(n - M)$ for $n = 0, 1, \ldots, 2M.$ ➡

$$b_0 = h(0-1) = h(-1) = 0.1871$$
$$b_1 = h(1-1) = h(0) = 0.2$$
$$b_2 = h(2-1) = h(1) = 0.1871$$

Example - Contd.

b. The transfer function: Causal FIR filter coefficients $H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots b_{2M} z^{-2M}$

where $b_n = h(n - M), n = 0, 1, \cdots, 2M.$

$$H(z) = 0.1871 + 0.2z^{-1} + 0.1871z^{-2} \Rightarrow \frac{Y(z)}{X(z)} = H(z) = 0.1871 + 0.2z^{-1} + 0.1871z^{-2} \Rightarrow Y(z) = 0.1871X(z) + 0.2z^{-1}X(z) + 0.1871z^{-2}X(z)$$

inverse z-transform ⇒ $\boxed{y(n) = 0.1871x(n) + 0.2x(n-1) + 0.1871x(n-2)}$

c. The magnitude frequency response and phase response

Substituting $z = e^{j\Omega}$ in $H(z)$ ⇒ $H(e^{j\Omega}) = 0.1871 + 0.2e^{-j\Omega} + 0.1871e^{-j2\Omega}$

Factoring term $e^{-j\Omega}$ ⇒ $H(e^{j\Omega}) = e^{-j\Omega}(0.1871e^{j\Omega} + 0.2 + 0.1871e^{-j\Omega})$

Using Euler formula $e^{jx} + e^{-jx} = 2\cos(x)$, ⇒ $H(e^{j\Omega}) = e^{-j\Omega}(0.2 + 0.3742\cos(\Omega))$

Magnitude and phase frequency response ⇒ $|H(e^{j\Omega})| = |0.2 + 0.3472\cos\Omega|$

⇒ $\angle H(e^{j\Omega}) = \begin{cases} -\Omega & \text{if } 0.2 + 0.3472\cos\Omega > 0 \\ -\Omega + \pi & \text{if } 0.2 + 0.3472\cos\Omega < 0 \end{cases}$

Ω radians	$f = \Omega f_s/2\pi$	$0.2+0.3742\cos\Omega$	$\lvert H(e^{j\Omega})\rvert$	$\lvert H(e^{j\Omega})\rvert_{dB}$	$\angle H(e^{j\Omega})$ degree
0	0	0.5742	0.5742	−4.82	0
$\pi/4$	1000	0.4646	0.4646	−6.66	−45
$\pi/2$	2000	0.2	0.2	−14.0	−90
$3\pi/4$	3000	−0.0646	0.0646	−23.8	45
π	4000	−0.1742	0.1742	−15.2	0

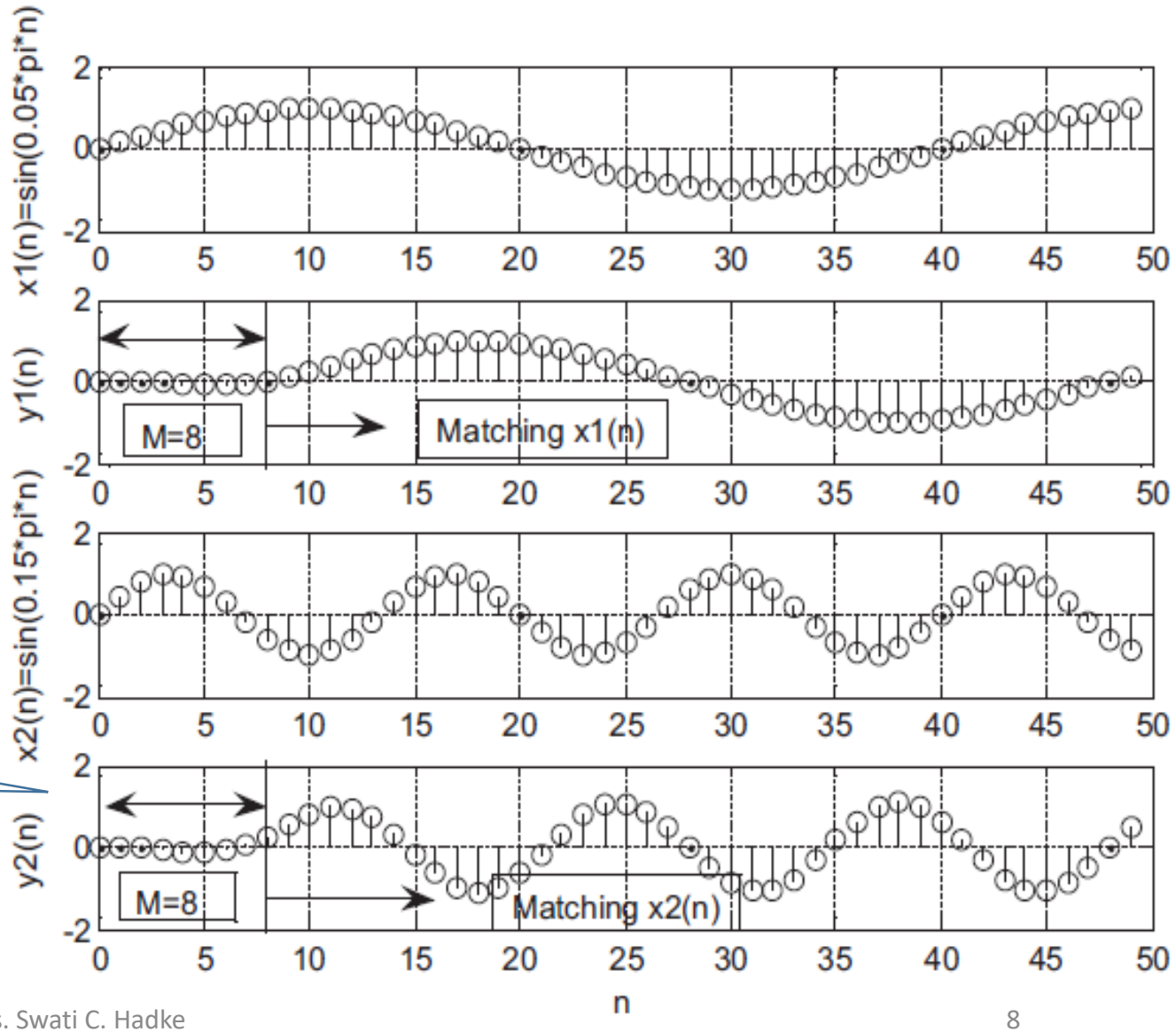
Linear Phase

If filter has linear phase property (the FIR coefficients are symmetric about the middle coefficient, and the FIR filter order is an odd number)**, the output will simply be a delayed version of input.**

M=8 samples delay

Let, 17-tap FIR filter with linear phase property (M=8).

M=8 samples delay



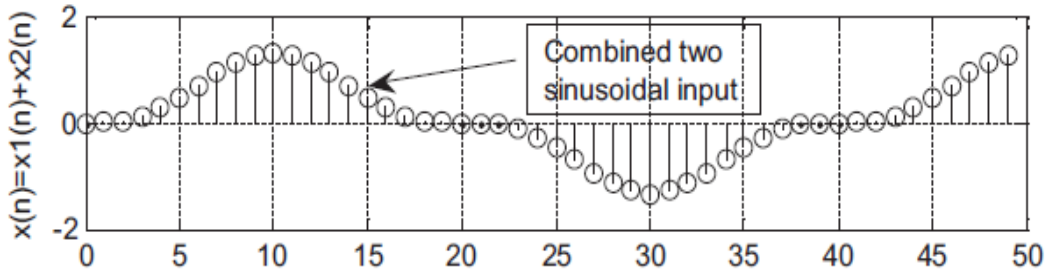
Nonlinear Phase

Input: $x(n) = x_1(n) + x_2(n) = \sin(0.05\pi n)u(n) - \frac{1}{3}\sin(0.15\pi n)u(n)$

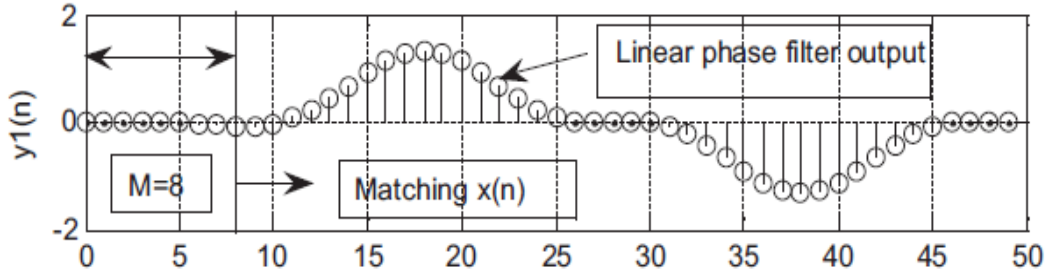
Linear phase filter output: $y_1(n) = \sin[0.05\pi(n-8)] - \frac{1}{3}\sin[0.15\pi(n-8)]$

90 degree Non-linear phase filter output: $y_2(n) = \sin(0.05\pi n - \pi/2) - \frac{1}{3}\sin(0.15\pi n - \pi/2)$

Input:

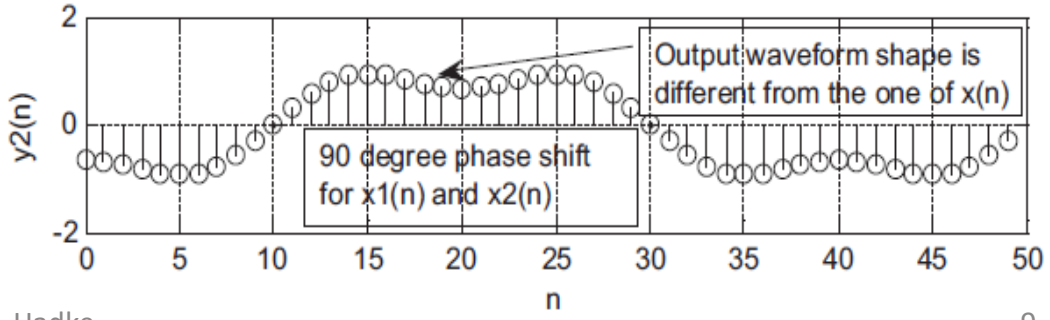


Linear phase filter output:



90 degrees phase delay filter output:

Distorted!



FIR Filtering With Window Method

- window functions are used to remedy the undesirable Gibbs oscillations.

- The window function, which is symmetrical and can gradually weight the designed FIR coefficients down to zeros at both ends for the range $-M \leq n \leq M$.

- Applying the window sequence $w(n)$ to the filter coefficients gives

$$h_w(n) = h(n) \cdot w(n)$$

1. Rectangular window:

$$w_{rec}(n) = 1, \ -M \leq n \leq M$$

2. Triangular (Bartlett) window:

$$w_{tri}(n) = 1 - \frac{|n|}{M}, \ -M \leq n \leq M$$

3. Hanning window:

$$w_{han}(n) = 0.5 + 0.5 \cos\left(\frac{n\pi}{M}\right), -M \leq n \leq M$$

4. Hamming window:

$$w_{ham}(n) = 0.54 + 0.46 \cos\left(\frac{n\pi}{M}\right), -M \leq n \leq M$$

5. Blackman window:

$$w_{black}(n) = 0.42 + 0.5 \cos\left(\frac{n\pi}{M}\right) + 0.08 \cos\left(\frac{2n\pi}{M}\right), \ -M \leq n \leq M$$

Example: Window Method

Problem:

Design a 5-tap FIR band reject (band-stop) filter with a lower cut-off frequency of 2,000 Hz, an upper cut-off frequency of 2,400 Hz, and a sampling rate of 8,000 Hz using the Hamming window method. Determine the transfer function.

Solution:

normalized cut-off frequencies

$$\Omega_L = 2\pi f_L T = 2\pi \times 2{,}000/8{,}000 = 0.5\pi \text{ radians}$$
$$\Omega_H = 2\pi f_H T = 2\pi \times 2{,}400/8{,}000 = 0.6\pi \text{ radians}$$

5-tap FIR → 2M + 1 = 5 then M=2 →

$$h(n) = \begin{cases} \dfrac{\pi - \Omega_H + \Omega_L}{\pi} & n = 0 \\ -\dfrac{\sin(\Omega_H n)}{n\pi} + \dfrac{\sin(\Omega_L n)}{n\pi} & n \neq 0 \quad -2 \le n \le 2 \end{cases}$$

$M = 2$

$$h(0) = \frac{\pi - \Omega_H + \Omega_L}{\pi} = \frac{\pi - 0.6\pi + 0.5\pi}{\pi} = 0.9$$

$$h(1) = \frac{\sin[0.5\pi \times 1]}{1 \times \pi} - \frac{\sin[0.6\pi \times 1]}{1 \times \pi} = 0.01558$$

$$h(2) = \frac{\sin[0.5\pi \times 2]}{2 \times \pi} - \frac{\sin[0.6\pi \times 2]}{2 \times \pi} = 0.09355$$

Symmetry

$$h(-1) = h(1) = 0.01558$$
$$h(-2) = h(2) = 0.09355$$

Example: Window Method –contd

Hamming window function

$$w_{ham}(0) = 0.54 + 0.46\cos\left(\frac{0\times\pi}{2}\right) = 1.0$$

$$w_{ham}(1) = 0.54 + 0.46\cos\left(\frac{1\times\pi}{2}\right) = 0.54$$

$$w_{ham}(2) = 0.54 + 0.46\cos\left(\frac{2\times\pi}{2}\right) = 0.08$$

Symmetry

$$w_{ham}(-1) = w_{ham}(1) = 0.54$$

$$w_{ham}(-2) = w_{ham}(2) = 0.08$$

Windowed impulse response

$$h_w(0) = h(0)w_{ham}(0) = 0.9\times 1 = 0.9$$

$$h_w(1) = h(1)w_{ham}(1) = 0.01558\times 0.54 = 0.00841$$

$$h_w(2) = h(2)w_{ham}(2) = 0.09355\times 0.08 = 0.00748$$

$$h_w(-1) = h(-1)w_{ham}(-1) = 0.00841$$

$$h_w(-2) = h(-2)w_{ham}(-2) = 0.00748$$

By delaying $h_w(n)$ by M = 2 samples, $b_0 = b_4 = 0.00748$, $b_1 = b_3 = 0.00841$, and $b_2 = 0.9$

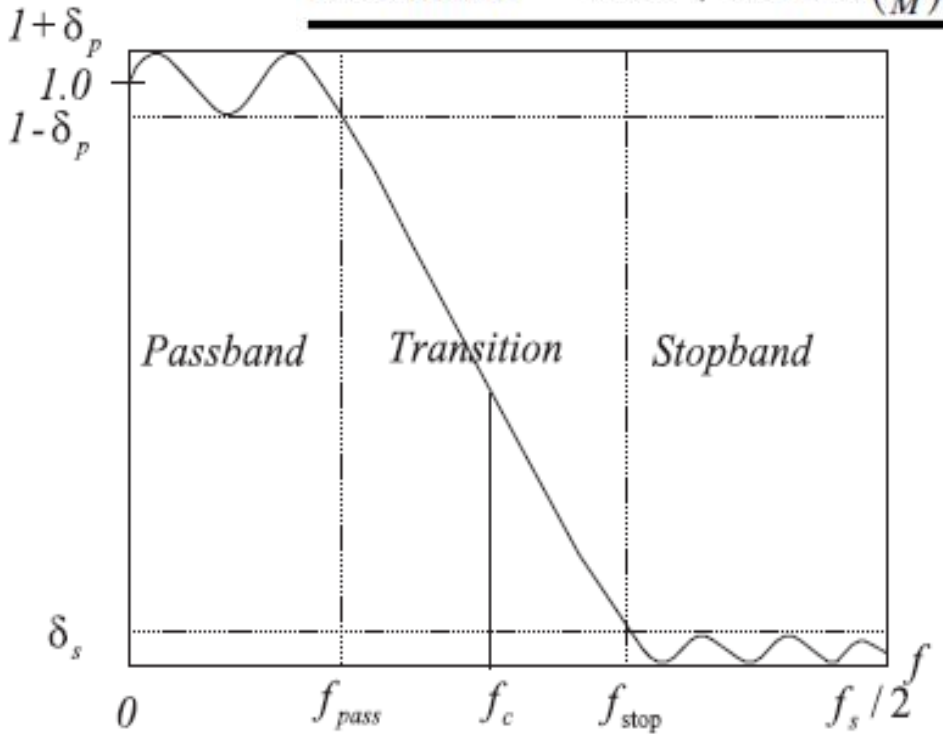
The transfer function

$$H(z) = 0.00748 + 0.00841z^{-1} + 0.9z^{-2} + 0.00841z^{-3} + 0.00748z^{-4}$$

FIR Filter Length Estimation

Given the required stop-band attenuation and pass-band ripple specifications the appropriate window can be selected.

Window Type	Window Function $w(n)$, $-M \le n \le M$	Window Length, N	Passband Ripple (dB)	Stopband Attenuation (dB)
Rectangular	1	$N = 0.9/\Delta f$	0.7416	21
Hanning	$0.5 + 0.5\cos\left(\frac{\pi n}{M}\right)$	$N = 3.1/\Delta f$	0.0546	44
Hamming	$0.54 + 0.46\cos\left(\frac{\pi n}{M}\right)$	$N = 3.3/\Delta f$	0.0194	53
Blackman	$0.42 + 0.5\cos\left(\frac{n\pi}{M}\right) + 0.08\cos\left(\frac{2n\pi}{M}\right)$	$N = 5.5/\Delta f$	0.0017	74



$$\Delta f = |f_{stop} - f_{pass}|/f_s$$

$$f_c = (f_{pass} + f_{stop})/2$$

$$\delta_p\ dB = 20 \cdot \log_{10}(1 + \delta_p)$$

$$\delta_s\ dB = -20\log_{10}(\delta_s)$$

Example: FIR Filter Length Estimation

Problem:

Design a BPF with

Lower stopband 0–500 Hz
Passband 1,600–2,300 Hz
Upper stopband 3,500–4,000 Hz
Stopband attenuation 50 dB
Passband ripple 0.05 dB
Sampling rate 8,000 Hz

Use Hamming window

Solution:

$\Delta f_1 = |1,600 - 500|/8,000 = 0.1375$
$\Delta f_2 = |3,500 - 2,300|/8,000 = 0.15$

$N_1 = 3.3/0.1375 = 24$
$N_2 = 3.3/0.15 = 22$

Choose nearest higher odd N = 25

Cutoff frequencies:

$f_1 = (1600 + 500)/2 = 1050\,\text{Hz}$
$f_2 = (3500 + 2300)/2 = 2900\,\text{Hz}.$

Normalized

$$\Omega_L = \frac{1050 \times 2\pi}{8000} = 0.2625\pi \text{ radians}$$

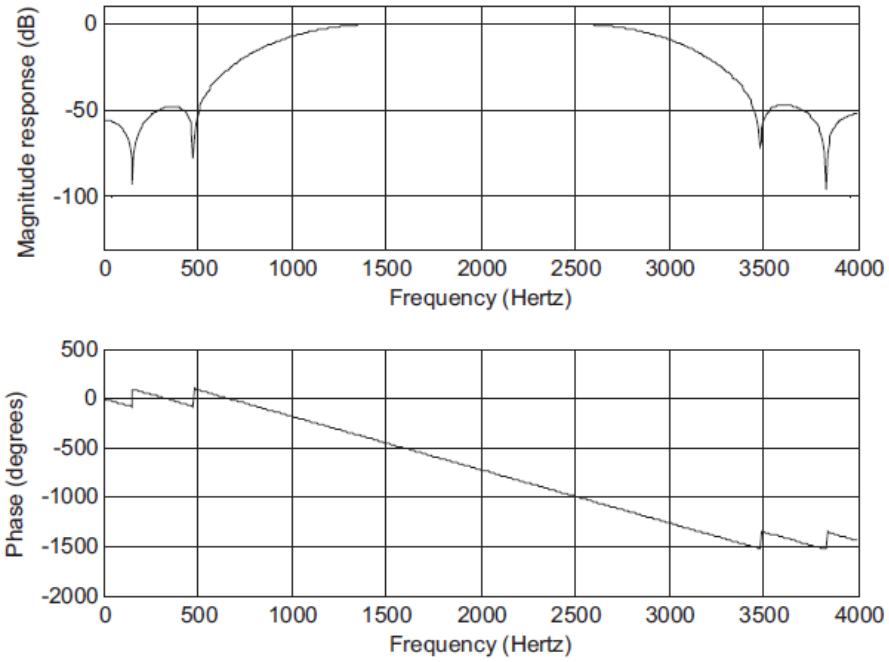
$$\Omega_H = \frac{2900 \times 2\pi}{8000} = 0.725\pi \text{ radians}$$

Using Matlab for the design

Filter type BPF

Window type Hamming

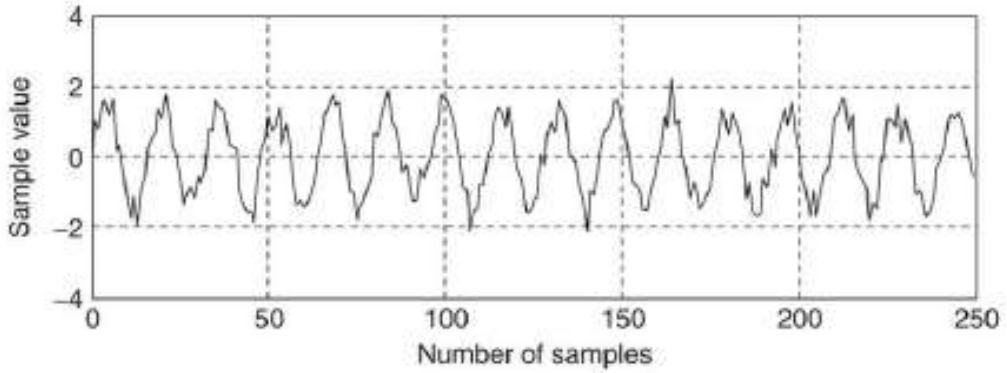
```
N=25; Ftype=3; WnL=0.2625*pi; WnH=0.725*pi; Wtype=4;fs=8000;
Bham=firwd(N,Ftype,WnL,WnH,Wtype);
freqz(Bham,1,512,fs);
axis([0 fs/2 -130 10]);
```



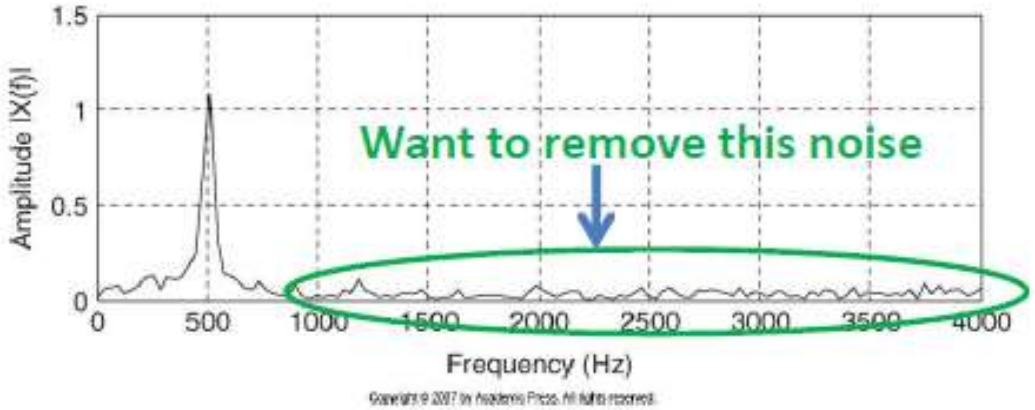
Application: Noise Reduction

We can design a digital filter to remove frequency components (noise) other than the desired frequency range.

Input waveform: sinusoid + broadband noise



Spectrum:



Specification: LPF

Pass band frequency [0 – 800 Hz]

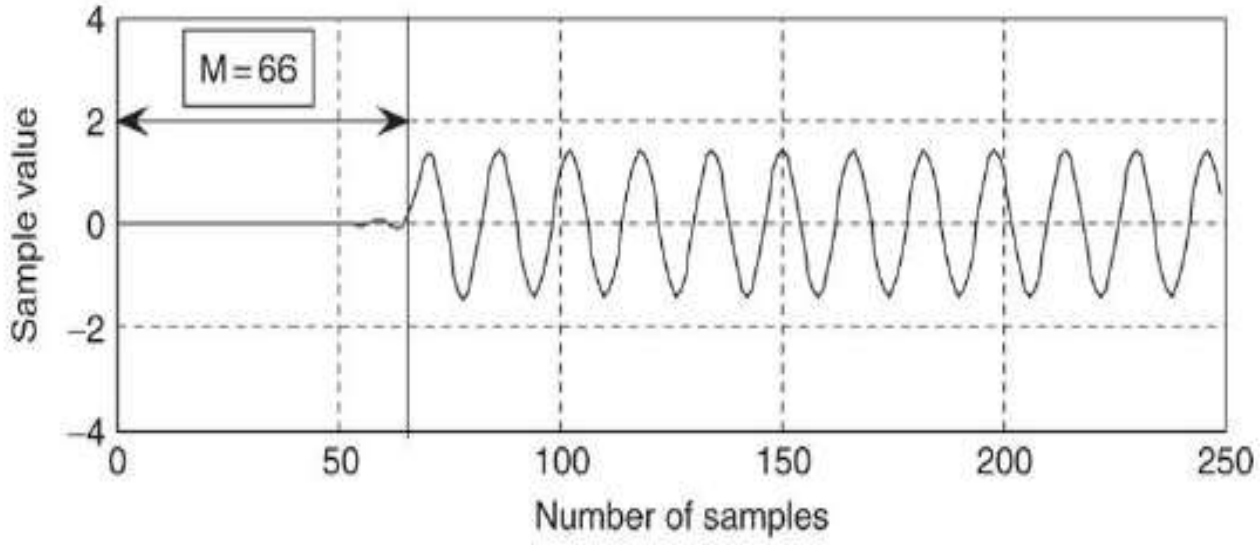
Stop band frequency [1000 – 4000 Hz]

Pass band ripple < 0.02 dB

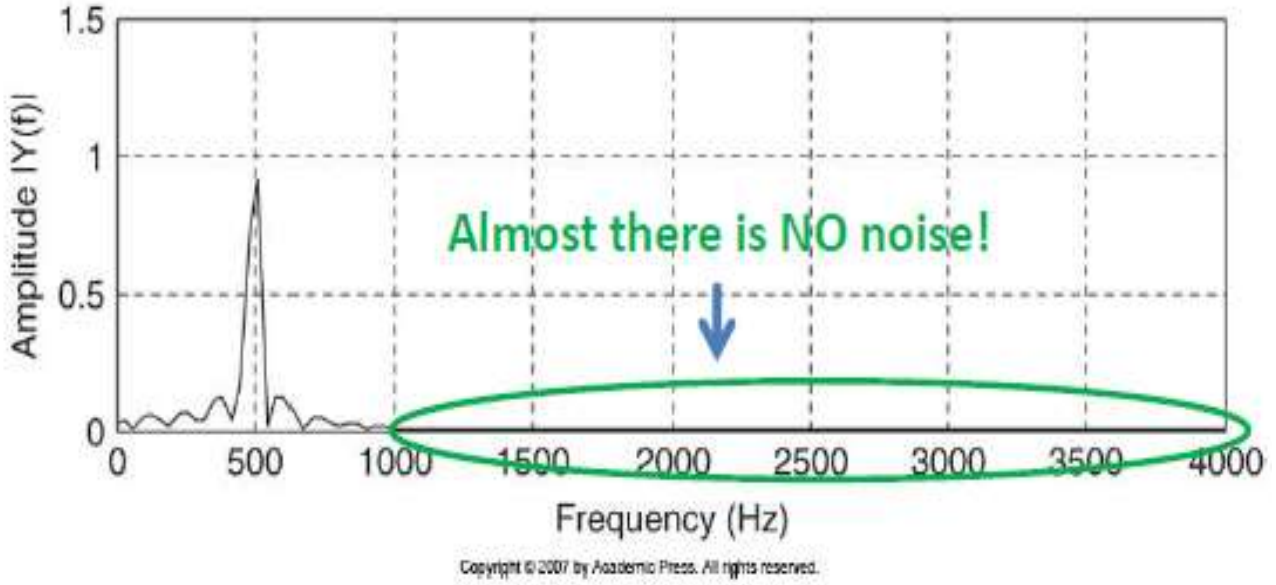
Stop band attenuation = 50 dB

Application: Noise Reduction -contd.

133- tap FIR filter, so a delay of 66



However, noise reduction in real world scenario is not so easy!



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Frequency Sampling Design Method

Simple to design

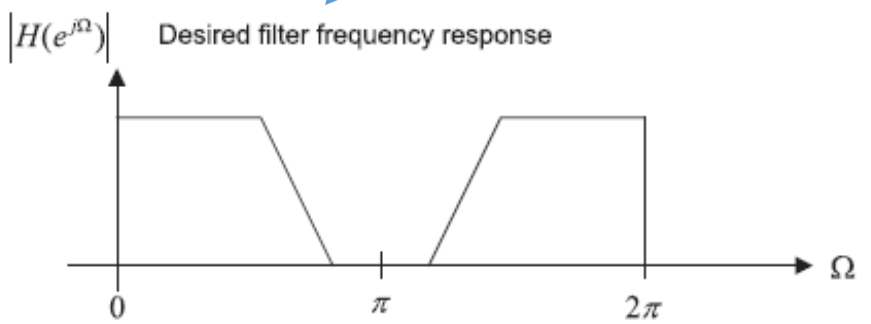
The key feature of frequency sampling is that the filter coefficients can be calculated based on the specified magnitudes of the desired filter frequency response uniformly in the frequency domain.

Filter length = 2M+1

$$H_k \text{ at } \Omega_k = \frac{2\pi k}{(2M+1)} \quad \text{for} \quad k = 0, 1, \cdots, M$$

H_k are obtained from sampled desired frequency response at equally spaced instants in frequency domain

Magnitude response in the range $[0, \ \pi]$

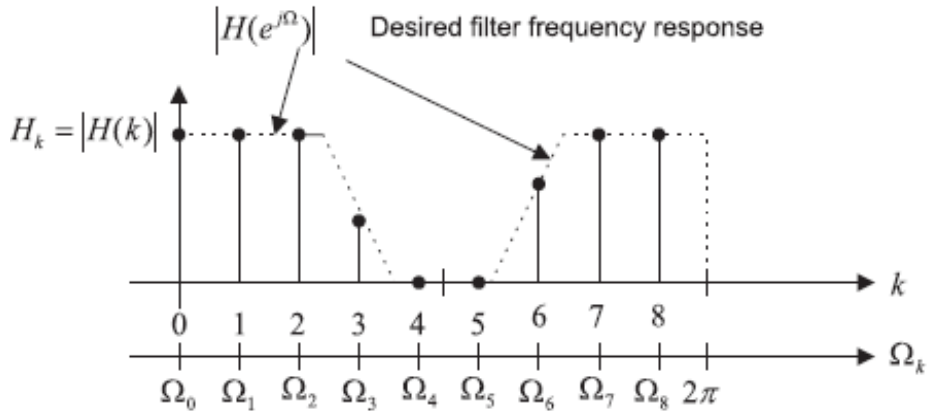


Calculate FIR filter coefficients:

$$h(n) = \frac{1}{2M+1}\left\{H_0 + 2\sum_{k=1}^{M} H_k \cos\left(\frac{2\pi k(n-M)}{2M+1}\right)\right\}, \quad \text{for} \quad n = 0, 1, \cdots, 2M$$

Use the symmetry:

$$h(n) = h(2M-n) \quad \text{for} \quad n = M+1, \cdots, 2M$$



Example: Frequency Sampling Design Method

Problem:

Design a linear phase low-pass FIR filter with 7 taps and a cut-off frequency of $\Omega_c = 0.3\pi$ radians using the frequency sampling method.

Solution:

$N = 2M + 1 = 7 \longrightarrow M = 3,$

Sampled frequencies

$$\Omega_k = \frac{2\pi}{7}k \text{ radians}, \; k = 0, 1, 2, 3$$

for $\Omega_0 = 0$ radians, $H_0 = 1.0$

for $\Omega_1 = \frac{2}{7}\pi$ radians, $H_1 = 1.0$

for $\Omega_2 = \frac{4}{7}\pi$ radians, $H_2 = 0.0$

for $\Omega_3 = \frac{6}{7}\pi$ radians, $H_3 = 0.0$

$$h(n) = \frac{1}{7}\left\{1 + 2\sum_{k=1}^{3} H_k \cos\left[2\pi k(n-3)/7\right]\right\}, \; n = 0, 1, \cdots, 3.$$

$$= \frac{1}{7}\{1 + 2\cos\left[2\pi(n-3)/7\right]\}$$

$h(0) = \frac{1}{7}\{1 + 2\cos(-6\pi/7)\} = -0.11456$

$h(1) = \frac{1}{7}\{1 + 2\cos(-4\pi/7)\} = 0.07928$

$h(2) = \frac{1}{7}\{1 + 2\cos(-2\pi/7)\} = 0.32100$

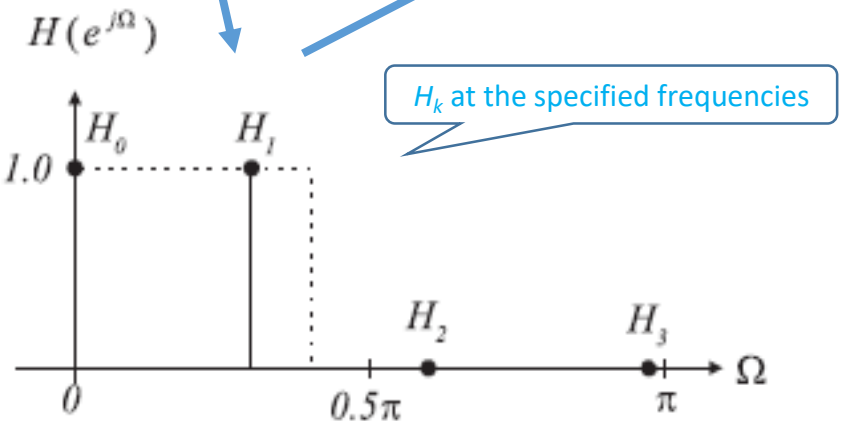
$h(3) = \frac{1}{7}\{1 + 2\cos(-0 \times \pi/7)\} = 0.42857$

By Symmetry

$h(4) = h(2) = 0.32100$

$h(5) = h(1) = 0.07928$

$h(6) = h(0) = -0.11456$



Coefficient Quantization Effect

- Obtaining filter coefficients with infinite precision is impossible.
- Filter coefficients are usually truncated or rounded off for the application.

FIR filter transfer function with infinite precision coefficients:

$$H(z) = \sum_{n=0}^{K} b_n z^{-n} = b_0 + b_1 z^{-1} + \cdots + b_{2M} z^{-K}$$

FIR filter transfer function with quantized precision

$$H^q(z) = \sum_{n=0}^{K} b_n^q z^{-n} = b_0^q + b_1^q z^{-1} + \cdots + b_K^q z^{-K}$$

Error of the magnitude frequency response

$$\left|H(e^{j\Omega}) - H^q(e^{j\Omega})\right| = \sum_{n=0}^{K} \left|(b_n - b_n^q)e^{j\Omega}\right|$$

K=tap

$$< \sum_{n=0}^{K} \left|b_n - b_n^q\right| < (K+1)\cdot 2^{-B}$$

B=Number of Bits
K=tap

Example

25-Tap FIR filter, 8 bit code (sign + 7 bits for fraction

Let infinite precision coeff. = 0.00759455135346

$0.00759455135346 \times 2^7 = 0.9721 \cong 1$ (rounded up to the integer)

Quantized coeff. = $1/2^7$ = 0.0078125

Error is bounded by (less than)

$$(K+1)2^{-B} = \frac{K+1}{2^B} = \frac{25}{2^7} = 0.0977$$

IIR Filter Design

Definition: Infinite impulse response (***IIR***) filter is described using :

IIR filter difference equation

$$y(n) = b_0x(n) + b_1x(n-1) + \cdots + b_Mx(n-M) - a_1y(n-1) - \cdots - a_Ny(n-N)$$

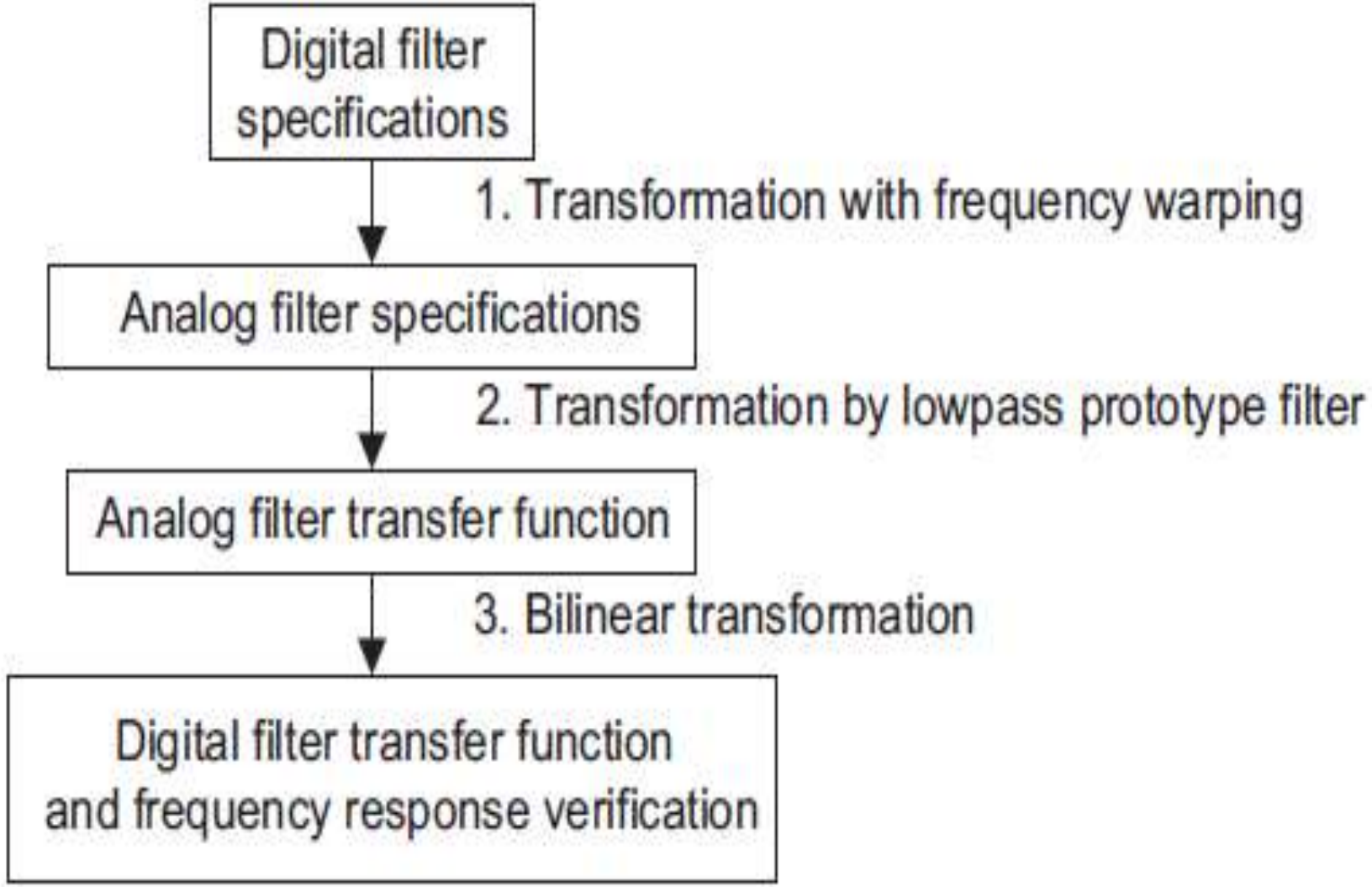
IIR filter output depends also on the past outputs

IIR filter transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1z^{-1} + \cdots + b_Mz^{-M}}{1 + a_1z^{-1} + \cdots + a_Nz^{-N}}$$

IIR Filter Design: Bilinear Transformation Method

Steps of the design procedure



Bilinear Transformation Method

Frequency Warping

For LPF and HPF:

$$\omega_a = \frac{2}{T}\tan\left(\frac{\omega_d T}{2}\right)$$

For BPF and BRF:

$$\omega_{al} = \frac{2}{T}\tan\left(\frac{\omega_l T}{2}\right),\ \omega_{ah} = \frac{2}{T}\tan\left(\frac{\omega_h T}{2}\right)$$

Obtained digital filter Transfer Function:

$$H(z) = H(s)\big|_{s=\frac{2}{T}\frac{z-1}{z+1}}$$

Prototype Transformation

From analog LPF to desired analog filter

$H_p(s)$

Low-pass prototype

$s \rightarrow \frac{s}{w_c}$ → $H_{LP}(s)$ — Analog Low-pass filter

w_c is the cut-off frequency

$s \rightarrow \frac{w_c}{s}$ → $H_{HP}(s)$ — Analog High-pass filter

w_c is the cut-off frequency

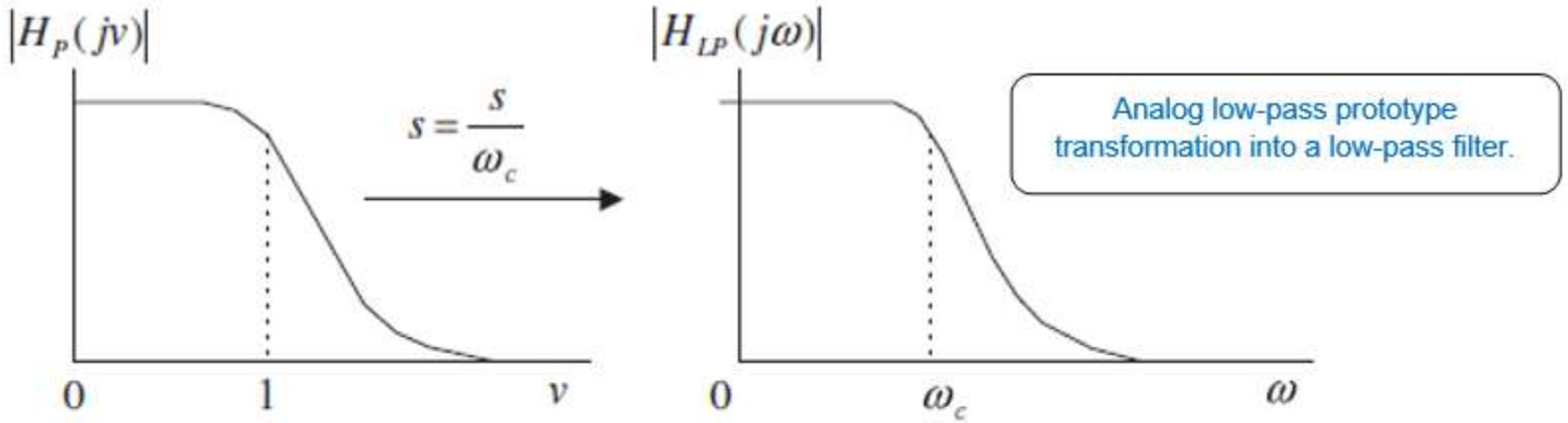
$s \rightarrow \frac{s^2 + \omega_0^2}{sW}$ → $H_{BP}(s)$ — Analog Band-pass filter

$w_0 = \sqrt{\omega_l \omega_h}, W = \omega_l - \omega_h$

$s \rightarrow \frac{sW}{s^2 + \omega_0^2}$ → $H_{BS}(s)$ — Analog Band-stop filter

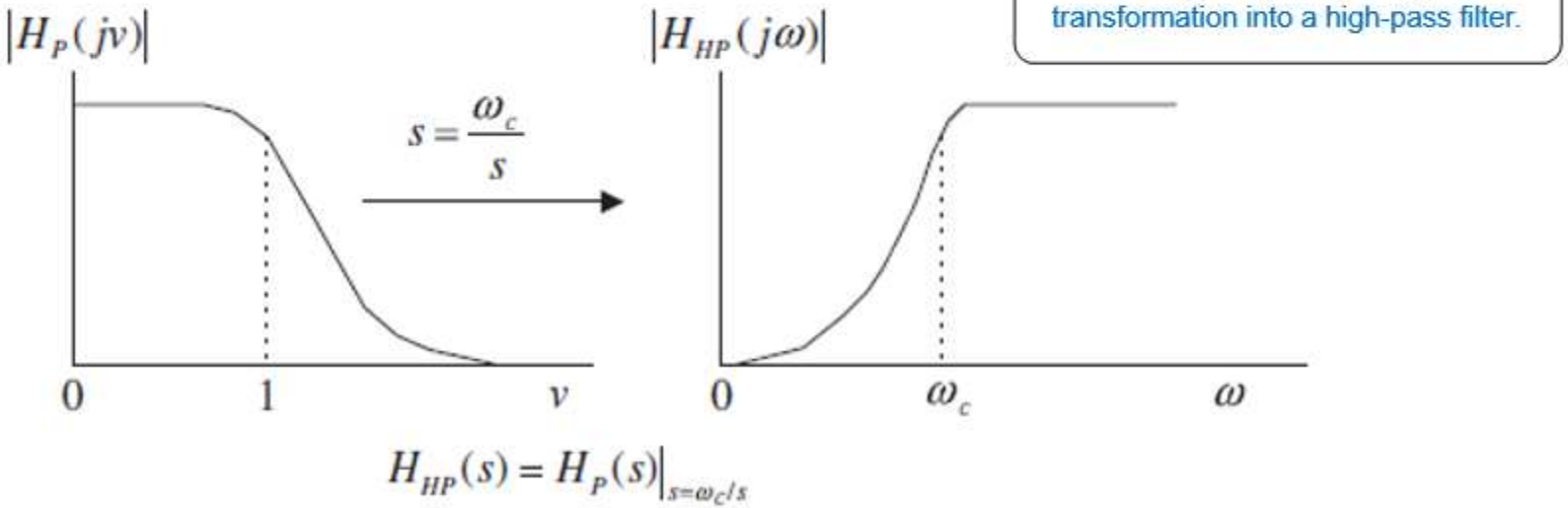
$w_0 = \sqrt{\omega_l \omega_h}, W = \omega_l - \omega_h$

Bilinear Transformation Method



Analog low-pass prototype transformation into a low-pass filter.

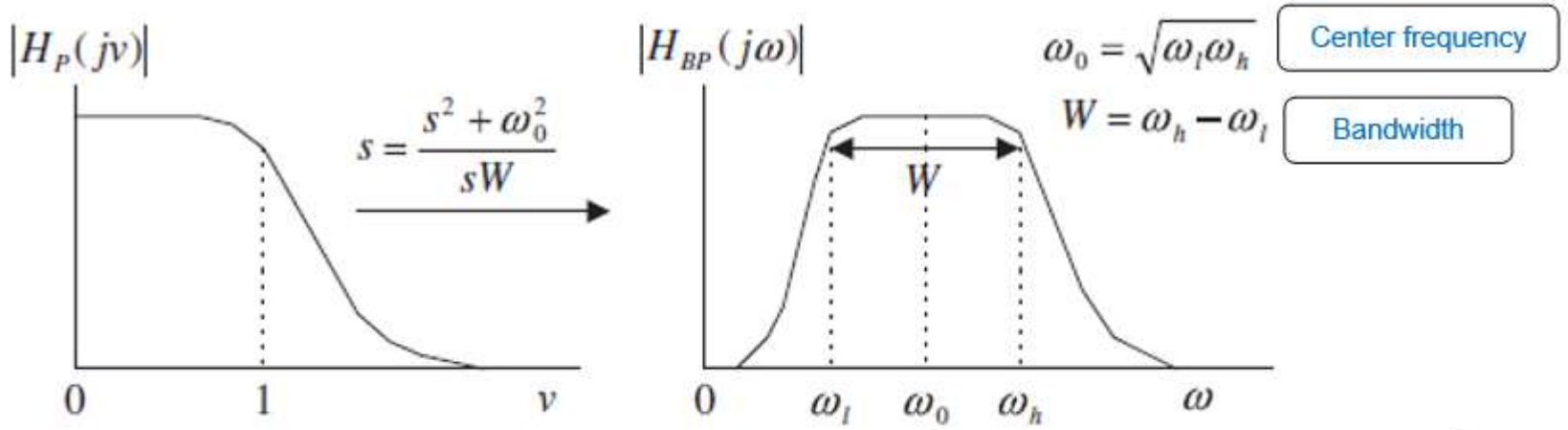
$$H_{LP}(s) = H_P(s)\Big|_{s=s/\omega_C}$$



Analog low-pass prototype transformation into a high-pass filter.

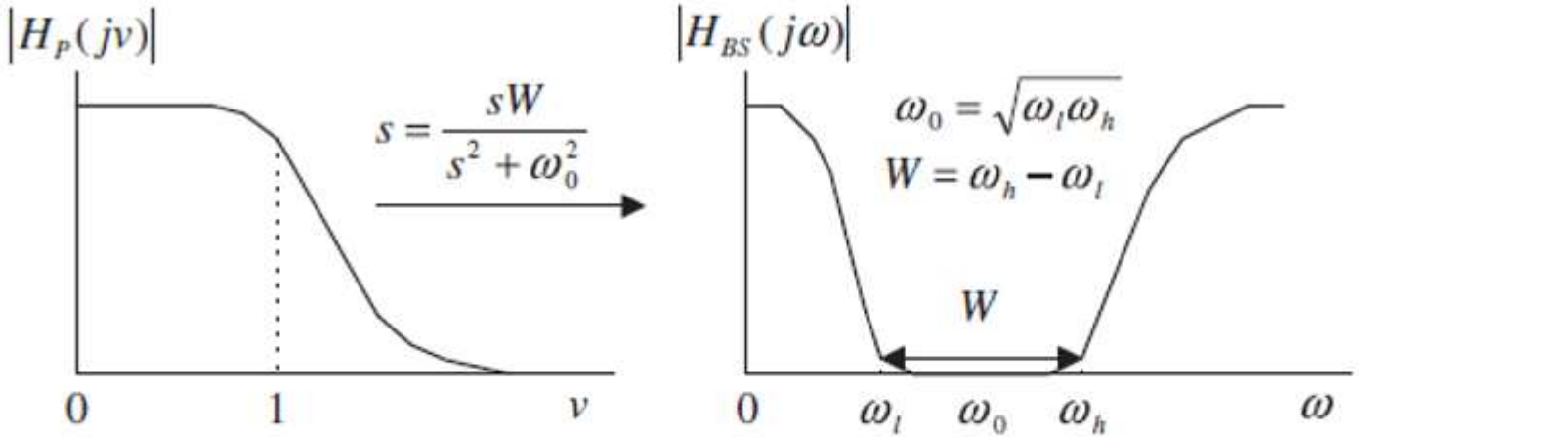
$$H_{HP}(s) = H_P(s)\Big|_{s=\omega_C/s}$$

Bilinear Transformation Method



$$H_{BP}(s) = H_P(s)\Big|_{s=\frac{s^2+\omega_0^2}{sW}}$$

Analog low-pass prototype transformation into a Band-pass filter.



$$H_{BS}(s) = H_P(s)\Big|_{s=\frac{sW}{s^2+\omega_0^2}}$$

Analog low-pass prototype transformation into a Band-stop filter.

Example 1: Bilinear Transformation Method

Problem: Design a first-order high-pass digital Chebyshev filter with a cut-off frequency of 3 kHz and 1 dB ripple on the pass-band using a sampling frequency of 8,000 Hz.

Solution:

Digital frequency (rad/s): $\omega_d = 2\pi f = 2\pi(3{,}000) = 6{,}000\pi \text{ rad/sec}, \quad \text{and} \quad T = 1/f_s = 1/8{,}000 \text{ sec}$

Pre-warped analog frequency : $\omega_a = \frac{2}{T}\tan\left(\frac{\omega_d T}{2}\right) = 16{,}000 \times \tan\left(\frac{6{,}000\pi/8{,}000}{2}\right) = 3.8627 \times 10^4 \text{ rad/sec}$

First-order LP Chebyshev filter prototype: $H_P(s) = \frac{1.9625}{s + 1.9625}$

Applying transformation LPF to HPF: $H(s) = H_P(s)\big|_{\frac{\omega_a}{s}} = \frac{1.9625}{\frac{\omega_a}{s} + 1.9625} = \frac{1.9625s}{1.9625s + 3.8627 \times 10^4}$

Dividing by 1.9625 → $H(s) = \frac{s}{s + 1.9683 \times 10^4}$

Applying BLT:

$$H(z) = \frac{s}{s + 1.9683 \times 10^4}\Big|_{s=16{,}000(z-1)/(z+1)} \Rightarrow H(z) = \frac{16{,}000\frac{z-1}{z+1}}{16{,}000\frac{z-1}{z+1} + 1.9683 \times 10^4} \Rightarrow H(z) = \frac{0.4484 - 0.4484z^{-1}}{1 + 0.1032z^{-1}}$$

Example 2: Bilinear Transformation Method

Problem:

Design a second-order digital band-pass Butterworth filter with the following specifications:

- upper cutoff frequency of 2.6 kHz
- lower cutoff frequency of 2.4 kHz
- sampling frequency of 8,000 Hz

Solution:

Digital frequencies:

$$\omega_h = 2\pi f_h = 2\pi(2600) = 5,200\pi \text{ rad/sec}$$
$$\omega_l = 2\pi f_l = 2\pi(2400) = 4,800\pi \text{ rad/sec, and } T = 1/f_s = 1/8,000 \text{ sec}$$

pre-warped analog frequency:

$$\omega_{ah} = \frac{2}{T}\tan\left(\frac{\omega_h T}{2}\right) = 16,000 \times \tan\left(\frac{5,200\pi/8,000}{2}\right) = 2.6110 \times 10^4 \text{ rad/sec}$$

$$\omega_{al} = 16,000 \times \tan\left(\frac{\omega_l T}{2}\right) = 16,000 \times \tan(0.3\pi) = 2.2022 \times 10^4 \text{ rad/sec}$$

$$W = \omega_{ah} - \omega_{al} = 26,110 - 22,022 = 4,088 \text{ rad/sec}$$

$$\omega_0^2 = \omega_{ah} \times \omega_{al} = 5.7499 \times 10^8$$

A first-order LPF prototype will produce second-order BPF prototype.

Example 2: Bilinear Transformation Method contd.

prototype transformation (LPF to BPF):

1st order LPF prototype: $H_P(s) = \frac{1}{s+1}$

Applying transformation LPF to BPF:

$$H(s) = H_P(s)\Big|_{\frac{s^2+\omega_0^2}{sW}} = \frac{Ws}{s^2+Ws+\omega_0^2} = \frac{4,088s}{s^2+4,088s+5.7499\times10^8}$$

Applying BLT:

$$H(z) = \frac{4,088s}{s^2+4,088s+5.7499\times10^8}\Big|_{s=16,000(z-1)/(z+1)}$$

$$H(z) = \frac{0.0730-0.0730z^{-2}}{1+0.7117z^{-1}+0.8541z^{-2}}$$

Pole Zero Placement Method

Second-Order BPF Design

r: controls bandwidth

θ: controls central frequency

Location of poles & zeros:

controls magnitude

Location of pole:

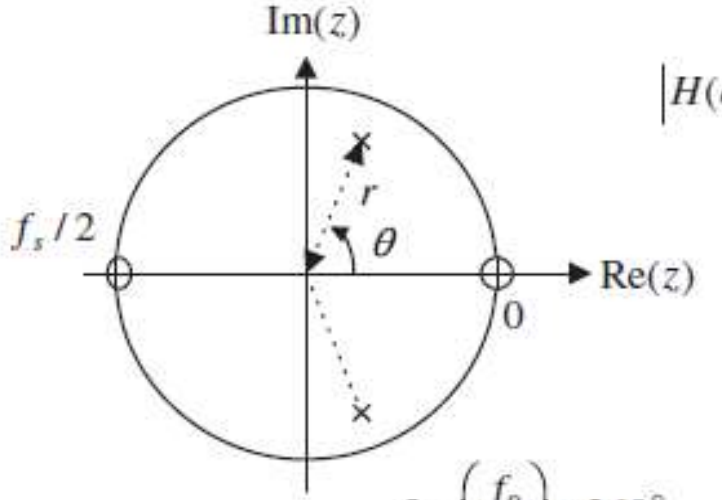
determines stability

Number of zero:

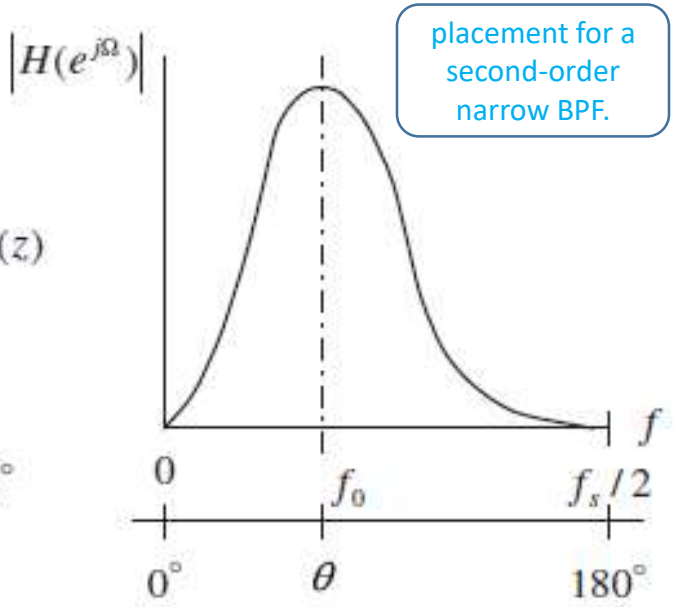
determines phase linearity

Poles are complex conjugate with magnitude ***r*** and the angle θ

Zeros are placed at z=1 corresponding to DC (zero frequency), and z= -1, corresponding to the folding frequency



$$\theta = \left(\frac{f_0}{f_s}\right) \times 360^\circ$$



placement for a second-order narrow BPF.

$$r \approx 1 - (BW_{3dB}/f_s) \times \pi, \text{ good for } 0.9 \le r < 1$$

$$\theta = \left(\frac{f_0}{f_s}\right) \times 360^\circ$$

$$K = \frac{(1-r)\sqrt{1 - 2r\cos 2\theta + r^2}}{2|\sin\theta|}$$

scale to have unit pass-band BPF gain

$$H(z) = \frac{K(z-1)(z+1)}{(z - re^{j\theta})(z - re^{-j\theta})} = \frac{K(z^2-1)}{(z^2 - 2rz\cos\theta + r^2)}$$

Pole Zero Placement Method

Second-Order BSF (Notch) Design

Poles amplitude Determines the 3 dB bandwidth

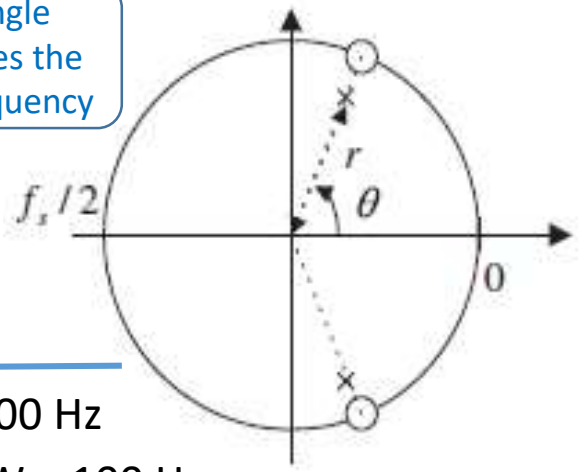
$$r \approx 1 - (BW_{3dB}/f_s) \times \pi, \text{ good for } 0.9 \le r < 1$$

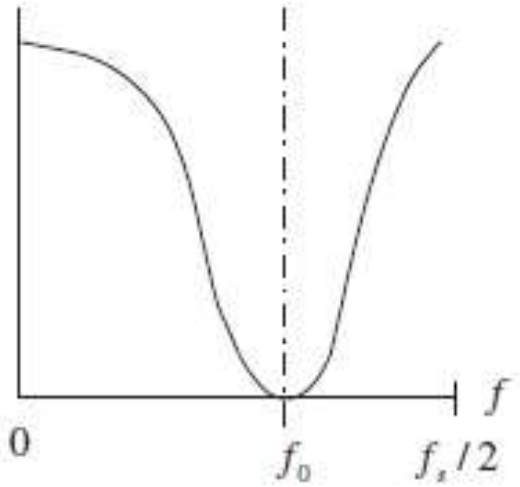
$$\theta = \left(\frac{f_0}{f_s}\right) \times 360^\circ$$

Poles angle Determines the center frequency

The zeros are placed on the unit circle with the same angles with respect to poles.

$$H(z) = \frac{K(z - e^{j\theta})(z + e^{-j\theta})}{(z - re^{j\theta})(z - re^{-j\theta})} = \frac{K(z^2 - 2z\cos\theta + 1)}{(z^2 - 2rz\cos\theta + r^2)}$$





$$K = \frac{(1 - 2r\cos\theta + r^2)}{(2 - 2\cos\theta)}$$

scale factor to adjust the BRF so it has a unit pass-band gain

Example

Design a second-order notch filter with

- Sampling rate = 8,000 Hz
- 3 dB bandwidth: BW = 100 Hz
- Narrow stop-band centered at f_0 = 1,500 Hz

$$r \approx 1 - (BW_{3dB}/f_s) \times \pi \Rightarrow r \approx 1 - (100/8,000) \times \pi = 0.9607$$

$$K = \frac{(1 - 2r\cos\theta + r^2)}{(2 - 2\cos\theta)} \Rightarrow K = \frac{(1 - 2 \times 0.9607\cos 67.5^\circ + 0.9607^2)}{(2 - 2\cos 67.5^\circ)} = 0.9620$$

$$\theta = \left(\frac{f_0}{f_s}\right) \times 360^\circ \Rightarrow \theta = \left(\frac{1,500}{8,000}\right) \times 360^\circ = 67.5^\circ$$

$$H(z) = \frac{K(z^2 - 2z\cos\theta + 1)}{(z^2 - 2rz\cos\theta + r^2)}$$

$$\Rightarrow H(z) = \frac{0.9620(z^2 - 2z\cos 67.5^\circ + 1)}{(z^2 - 2 \times 0.9607z\cos 67.5^\circ + 0.9607^2)}$$

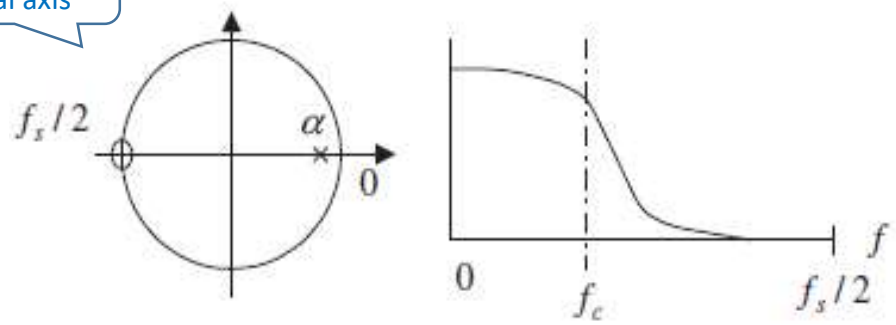
$$\Rightarrow H(z) = \frac{0.9620 - 0.7363z^{-1} + 0.9620z^{-2}}{1 - 0.7353z^{-1} + 0.9229}$$

Pole Zero Placement Method

First-Order LPF Design

When Cut-off frequency $f_c < {}^{f_s}/_4$

Pole placed on the real axis

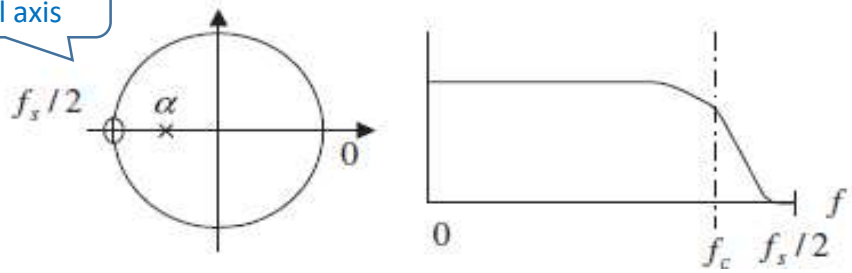


$$\alpha \approx 1 - 2 \times (f_c/f_s) \times \pi, \quad \text{good for} \quad 0.9 \le r < 1$$

The zero is placed at $(z = -1)$

When Cut-off frequency $f_c > {}^{f_s}/_4$

Pole placed on the real axis



$$\alpha \approx -(1 - \pi + 2 \times (f_c/f_s) \times \pi), \quad \text{good for} \quad -1 < r \le -0.9$$

Transfer function

$$H(z) = \frac{K(z+1)}{(z-\alpha)}$$

$$K = \frac{(1-\alpha)}{2}$$

unit pass-band gain scale factor

Example

transfer function of first-order LPF is required to satisfy:

- Sampling rate = 8,000 Hz
- 3 dB cutoff frequency: f_c = 100 Hz
- Zero gain at 4,000 Hz

Since cut-off frequency = 100 Hz < $\frac{f_s}{4} = 2000Hz$, pole at:

$$\alpha \approx 1 - 2 \times (f_c/f_s) \times \pi$$

$$\alpha \approx 1 - 2 \times (100/8,000) \times \pi = 0.9215$$

unit gain scale factor: $K = \frac{(1-\alpha)}{2}$

$$K = \frac{(1-0.9215)}{2} = 0.03925$$

transfer function: $H(z) = \frac{K(z+1)}{(z-\alpha)}$

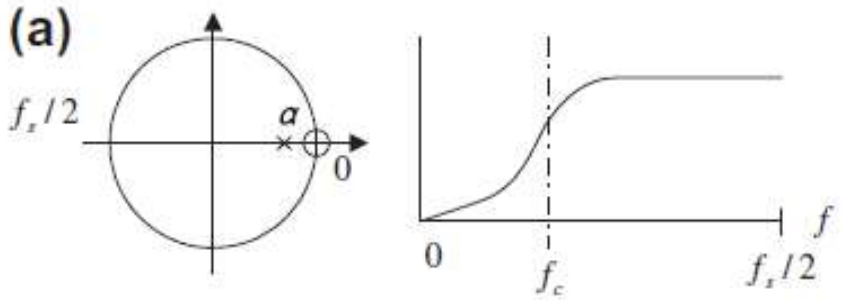
$$H(z) = \frac{0.03925(z+1)}{(z-0.9215)} = \frac{0.03925 + 0.03925z^{-1}}{1 - 0.9215z^{-1}}$$

Pole Zero Placement Method

First-Order HPF Design

When Cut-off frequency $f_c < {}^{f_s}/_4$

Pole placed on the real axis

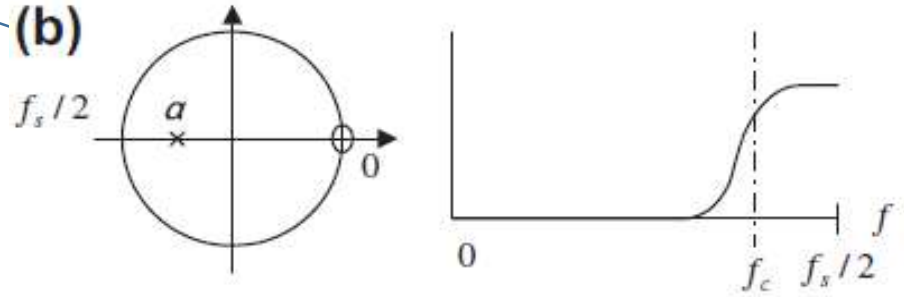


$$\alpha \approx 1 - 2 \times (f_c/f_s) \times \pi, \quad \text{good for} \quad 0.9 \leq r < 1$$

The zero is placed at $(z = +1)$

When Cut-off frequency $f_c > {}^{f_s}/_4$

Pole placed on the real axis



$$\alpha \approx -(1 - \pi + 2 \times (f_c/f_s) \times \pi), \quad \text{good for} \quad -1 < r \leq -0.9$$

Transfer function

$$H(z) = \frac{K(z-1)}{(z-\alpha)}$$

$$K = \frac{(1+\alpha)}{2}$$

unit pass-band gain scale factor

Example

transfer function of first-order HPF is required to satisfy:

- Sampling rate = 8,000 Hz
- 3 dB cutoff frequency: $f_c = 3,800$ Hz
- Zero gain at 0 Hz

Since cut-off frequency = 3800 Hz > $\frac{f_s}{4} = 2000Hz$, pole at

$$\alpha \approx -(1 - \pi + 2 \times (f_c/f_s) \times \pi)$$

$$\alpha \approx -(1 - \pi + 2 \times (3,800/8,000) \times \pi) = -0.8429$$

unit gain scale factor: $K = \frac{(1-\alpha)}{2}$

$$K = \frac{(1 - 0.8429)}{2} = 0.07854$$

transfer function: $H(z) = \frac{K(z-1)}{(z-\alpha)}$

$$H(z) = \frac{0.07854(z-1)}{(z+0.8429)} = \frac{0.07854 - 0.07854z^{-1}}{1 + 0.8429z^{-1}}$$

IIR Filter realization

Example: Realize the first-order digital high-pass Butterworth filter using a direct-form I realization.

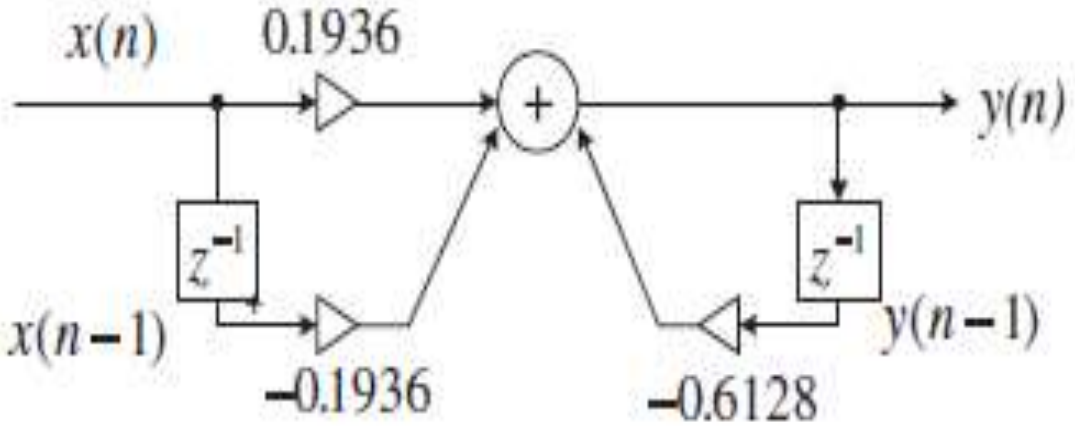
$$H(z) = \frac{0.1936 - 0.1936z^{-1}}{1 + 0.6128z^{-1}}$$

Solution:

Filter coefficients: $b_0 = 0.1936,\ b_1 = -0.1936,\ \text{and}\ \ a_1 = 0.6128$

DSP equation: $y(n) = -0.6128y(n-1) + 0.1936x(n) - 0.1936x(n-1)$

Implementation:

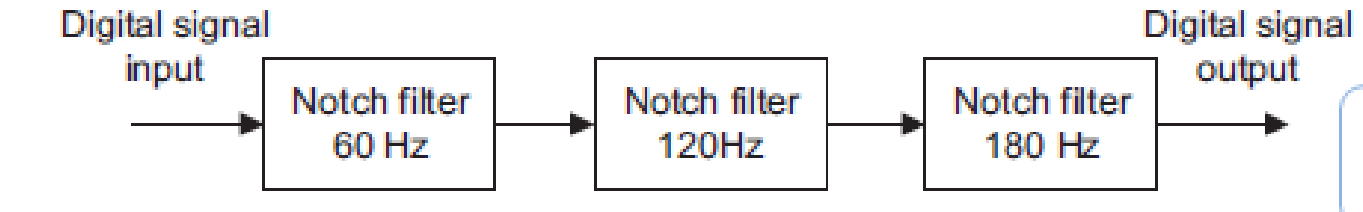


Application1: 60 –Hz Hum Eliminator

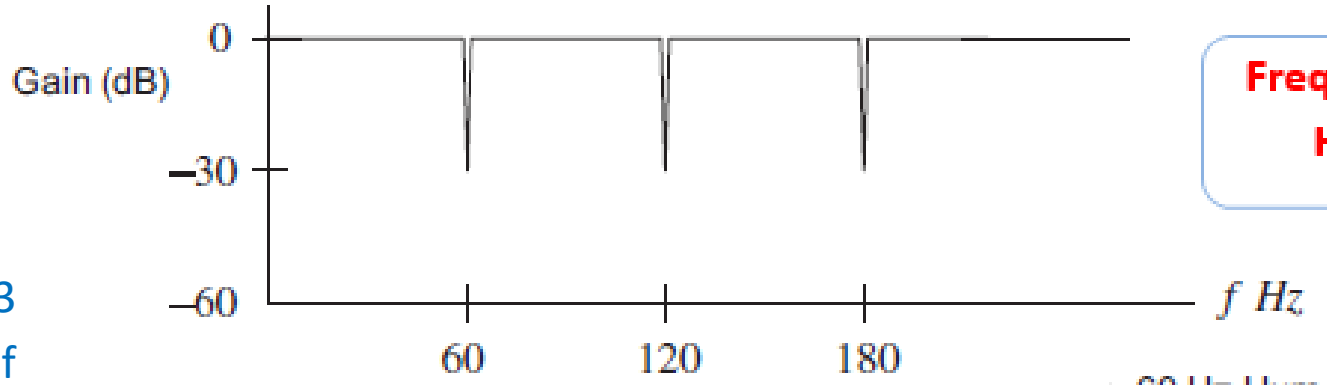
***Hum noise*: created by poor power supply or electromagnetic interference and characterized by a frequency of 60 Hz and its harmonics.**

Eliminate the 60-Hz hum frequency with its second and third harmonics in most practical applications.

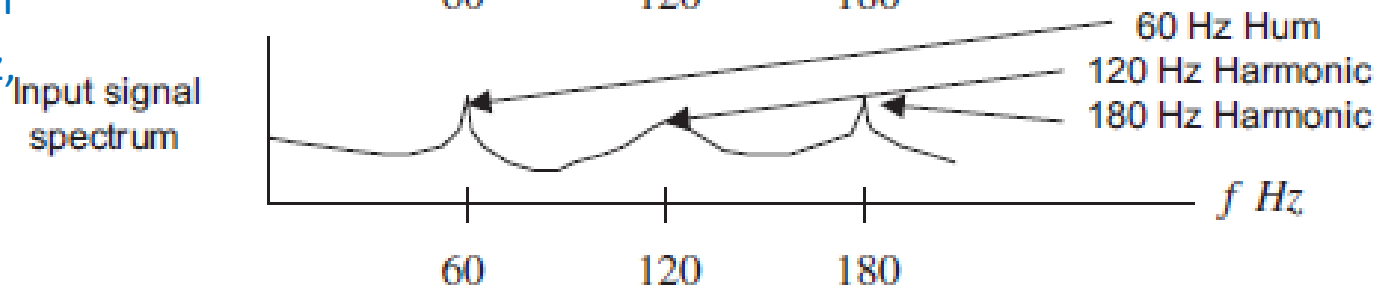
We can do this by cascading 3 notch filters having frequencies of 60 Hz, 120 Hz, and 180 Hz, respectively.



Hum eliminator



Frequency response of Hum eliminator



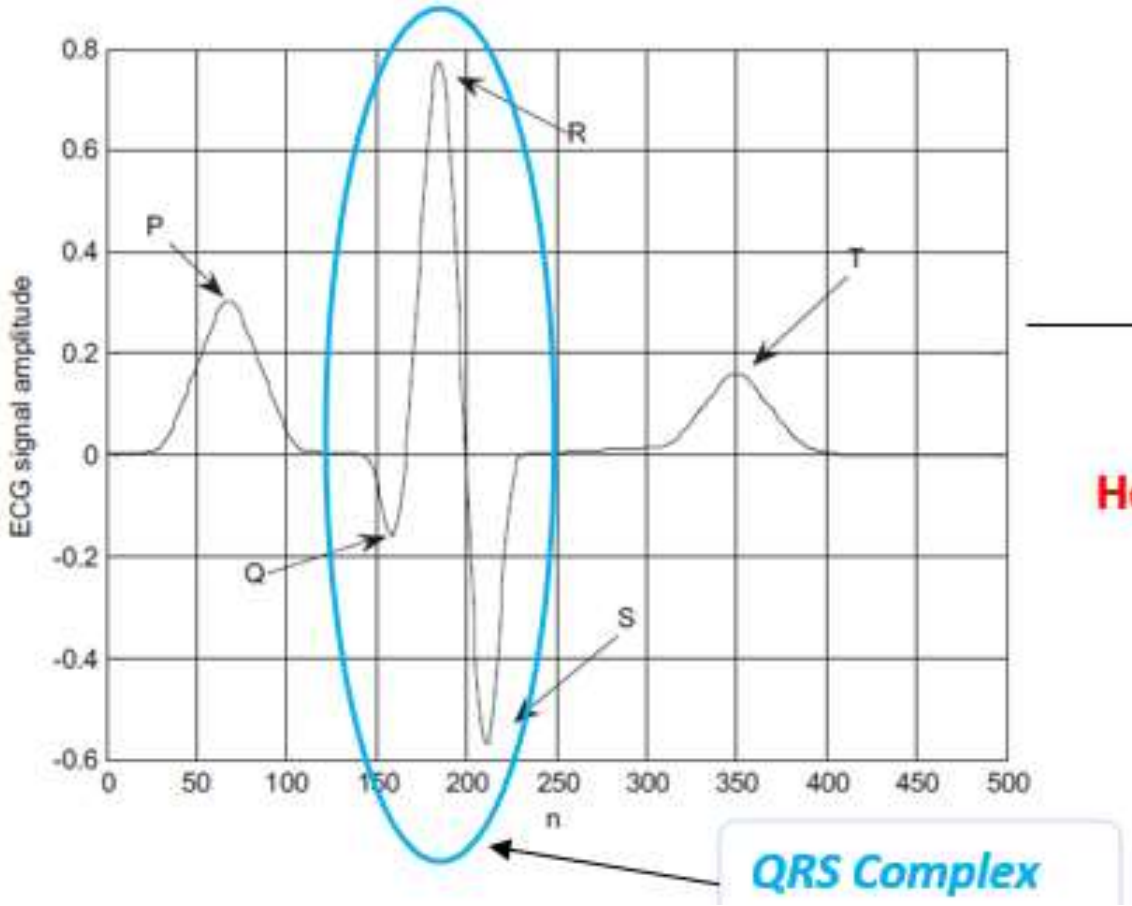
Corrupted by hum and harmonics

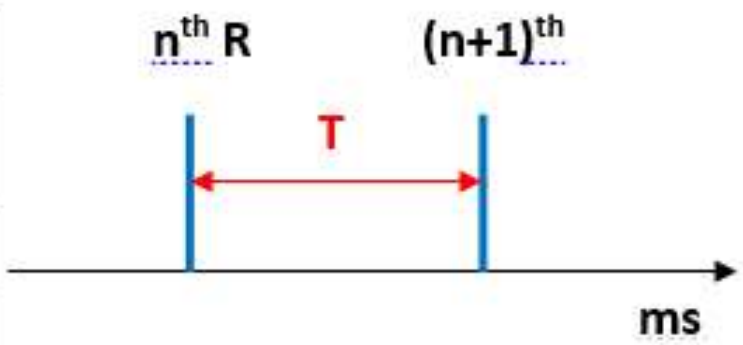
Application2: ECG Pulse

The ECG signal is produced by the electrical activity of the human heart, it is characterized by five peaks and valleys ***P, Q, R, S***, and ***T***.

The properties of the ***QRS*** complex, (rate of occurrence and times, heights, and widths, provide information to cardiologists concerning various pathological conditions of the heart.

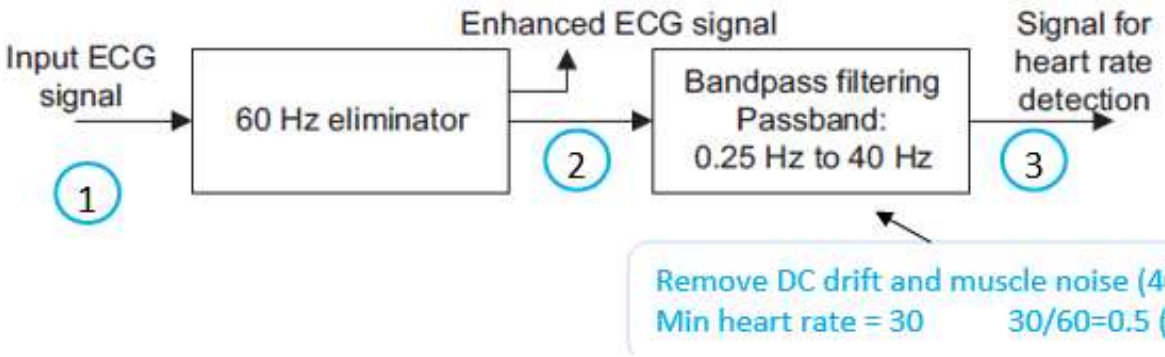
ECG + Hum ⟶ makes difficult to analyze.



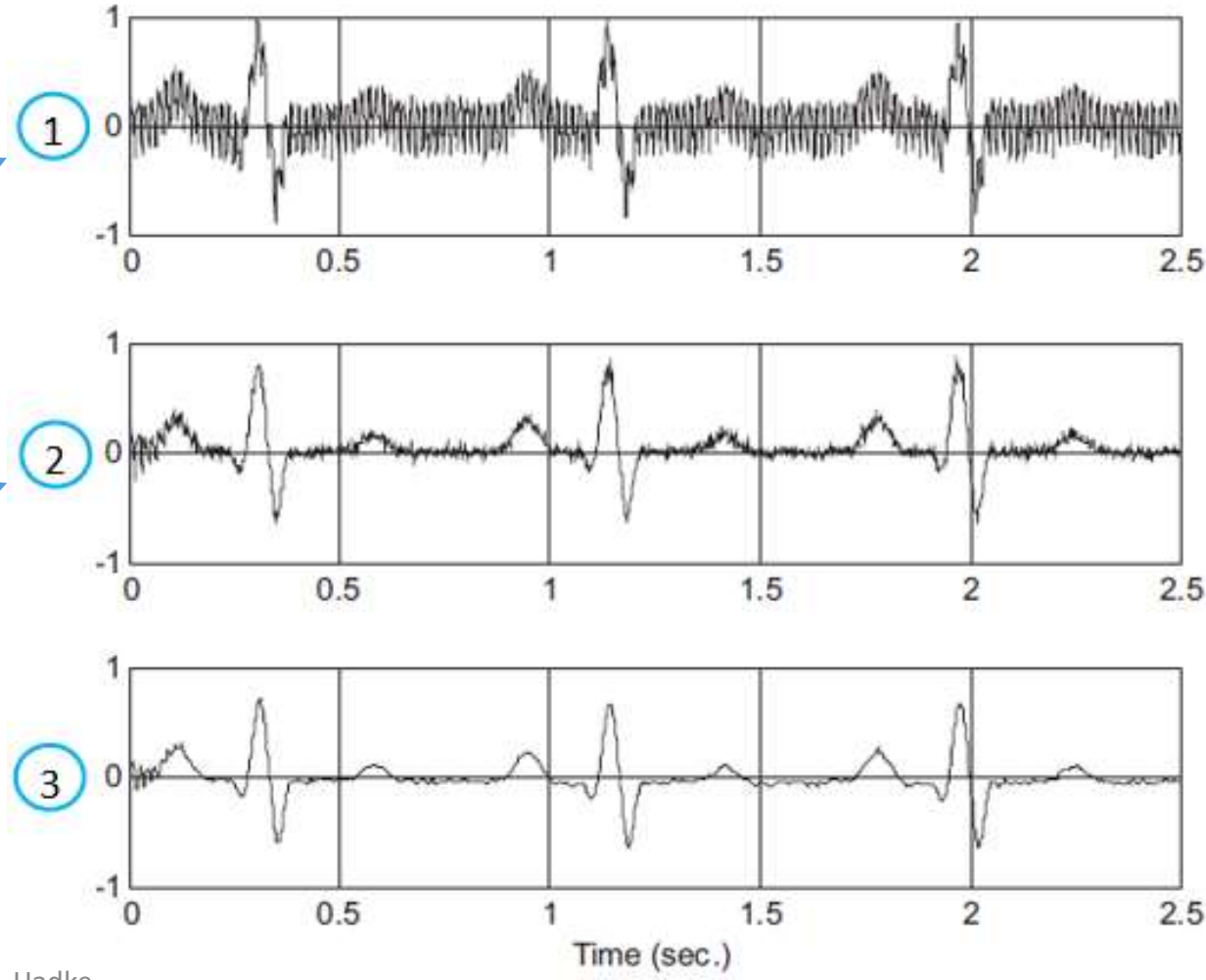


Heart beat/min= 60000/T

Heart Beat Detection Using ECG Pulse

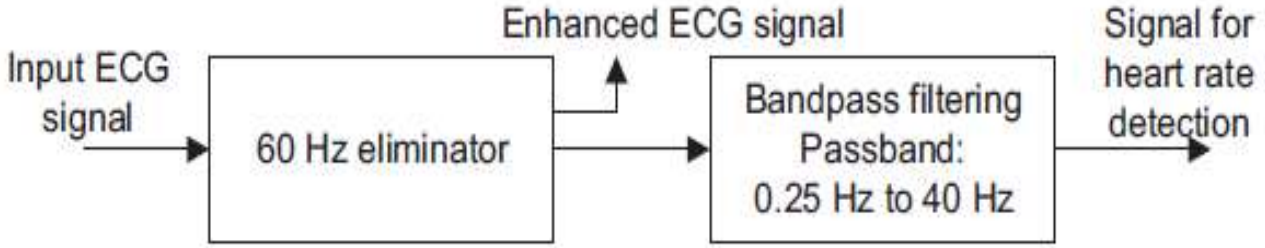


- Initial corrupted ECG data (with 60-Hz interference and its 120 and 180 Hz harmonics, and muscle noise).
- The 60-Hz interference and its harmonics of 120 and 180 Hz have been removed
- Result after the band-pass filter
- Original ECG signal occupies the frequency range from 0.01 Hz to 250 Hz (used for diagnostic-quality ECG).



Example: EGC Signal pre-processing

Problem: Design the ECG signal pre-processing system for heart beat rate detection



Solution:

60-Hz eliminator design (notch frequency of 60Hz)

60-Hz eliminator specifications:

Harmonics to be removed: 60 Hz (fundamental)
120 Hz (second harmonic)
180 Hz (third harmonic)

3-dB bandwidth for each filter: 4 Hz
Sampling rate: 600 Hz
Notch filter type: Second-order IIR

$$r = 1 - (4/600) \times \pi = 0.9791$$

$$\theta = \left(\frac{60}{600}\right) \times 360^\circ = 36^\circ$$

$$K = \frac{(1 - 2r\cos\theta + r^2)}{(2 - 2\cos\theta)} = 0.9803$$

Example: EGC Signal pre-processing -contd

transfer functions and the difference equations

$$H_1(z) = \frac{0.9803 - 1.5862z^{-1} + 0.9803z^{-2}}{1 - 1.5842z^{-1} + 0.9586z^{-2}}$$

$$y_1(n) = 0.9803x(n) - 1.5862x(n-1) + 0.9802x(n-2) + 1.5842y_1(n-1) - 0.9586y_1(n-2)$$

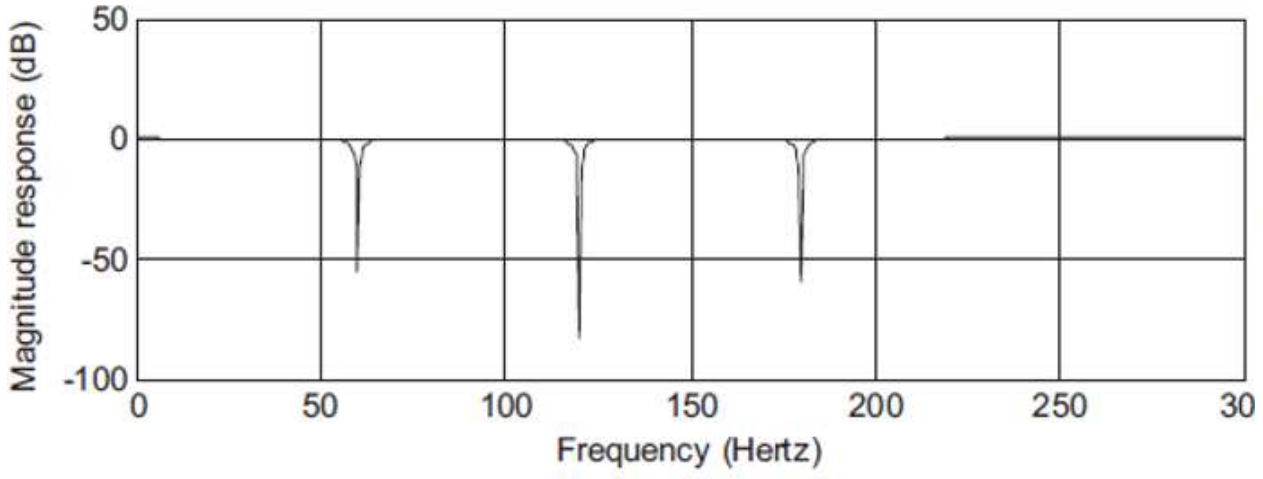
$$H_2(z) = \frac{0.9794 - 0.6053z^{-1} + 0.9794z^{-2}}{1 - 0.6051z^{-1} + 0.9586z^{-2}}$$

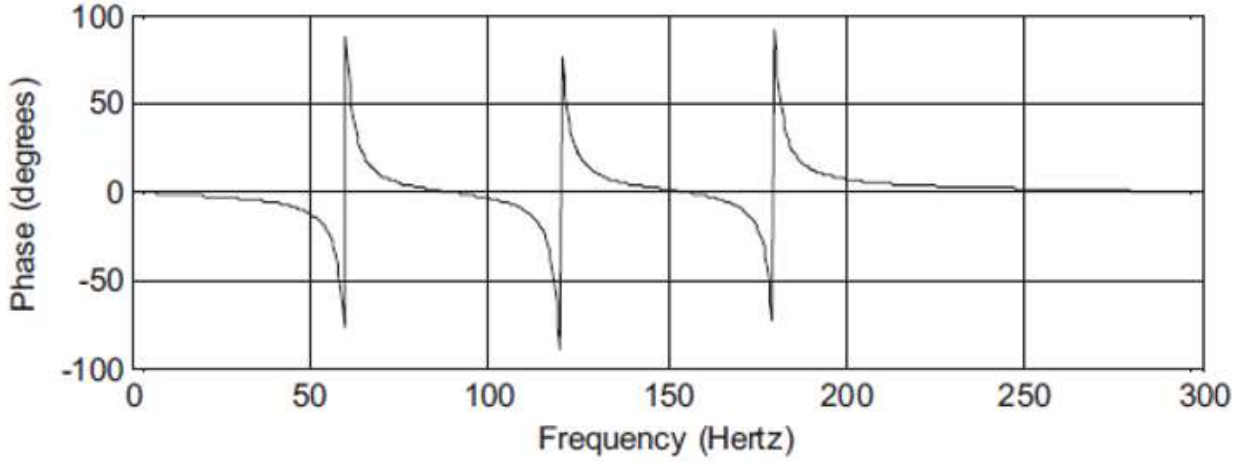
$$y_2(n) = 0.9794y_1(n) - 0.6053y_1(n-1) + 0.9794y_1(n-2) + 0.6051y_2(n-1) - 0.9586y_2(n-2)$$

$$H_3(z) = \frac{0.9793 + 0.6052z^{-1} + 0.9793z^{-2}}{1 + 0.6051z^{-1} + 0.9586z^{-2}}$$

$$y_3(n) = 0.9793y_2(n) + 0.6052y_2(n-1) + 0.9793y_2(n-2) - 0.6051y_3(n-1) - 0.9586y_3(n-2)$$

Frequency responses of three cascaded notch filters.





Example: EGC Signal pre-processing -contd

The second-stage design using the BLT

Design method:	Pole-zero placement
Bandpass filter specifications:	
Passband frequency range:	0.25–40 Hz
Passband ripple:	0.5 dB
Filter type:	Chebyshev fourth order
Design method:	Bilinear transformation method
DSP sampling rate:	600 Hz

$$H_4(z) = \frac{0.0464 - 0.0927z^{-2} + 0.0464z^{-4}}{1 - 3.3523z^{-1} + 4.2557z^{-2} - 2.4540z^{-3} + 0.5506z^{-4}}$$

$$y_4(n) = 0.046361y_3(n) - 0.092722y_3(n-2) + 0.046361y_3(n-4)$$
$$+03.352292y_4(n-1) - 4.255671y_4(n-2) + 2.453965y_4(n-3) - 0.550587y_4(n-4)$$

